

タイトル	ガウス型定常確率過程における過去・現在・未来
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ガウス型定常確率過程における 過去・現在・未来

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1. 序 文

平均が0であるような確率変数の集合 G の任意の元 X_1, \dots, X_n について, その1次結合 Y が Gauss 分布:

$$P(a \leq Y < b) = \frac{1}{\sqrt{2\pi V}} \int_a^b e^{-y^2/2V} dy$$

を満足するとき, G は Gauss 属であるという。(cf. H. Dym and H.P. McKean [17]) ただし, $V = E(Y^2)$ である。このとき,

$$\|Y\| = \{E(Y^2)\}^{1/2}$$

として G に Hilbert 空間の構造を導入し, そのようにしてできた Hilbert 空間を同じ記号 G で表す。 G はそれ自身 Gauss 属であるし, その部分空間達も Gauss 属である。 G の2つの部分空間 A と B の間の独立性を測る量として, A と B がなす角の cosine:

$$c_1 = \text{supremum of } E(XY) \text{ for } X \in A \text{ and } Y \in B \text{ of length } \leq 1$$

がある。Gauss 属 G の任意の元が時間のパラメータ t により $X(t)$ と表されているとする。もし

$$P(\bigcap_{i=1}^n (a_i \leq X(t_i + T) < b_i))$$

が T に依存しないならば, G はガウス型定常確率過程と呼ばれる。以下では, G のスペクトル関数が W である場合を考える。

$L^2(W)$ から $H^2(W)$ への射影作用素を P で表し, $Q = I - P$ と定める。このとき,

$$P\left(\sum_{n=-\infty}^{\infty} a_n z^n\right) = \sum_{n=0}^{\infty} a_n z^n,$$
$$Q\left(\sum_{n=-\infty}^{\infty} a_n z^n\right) = \sum_{n=-\infty}^{-1} a_n z^n$$

が成り立つ。更に、 $L^2(W)$ から $L^2(W)$ へのシフト作用素を U で表す。このとき、

$$U\left(\sum_{n=-\infty}^{\infty} a_n z^n\right) = \sum_{n=-\infty}^{\infty} a_n z^{n+1}$$

が成り立つ。このとき、 $S=U|_{H^2(W)}$ と定める。

$$S\left(\sum_{n=0}^{\infty} a_n z^n\right) = \sum_{n=0}^{\infty} a_n z^{n+1}$$

が成り立つ。

特に、 W が恒等的に 1 に等しい定数関数のとき、 $H^2(W)=H^2, L^2(W)=L^2$ が成り立つ。

Beurling の定理 次の等式がなりたつ。

$$\{E \subset H^2: E \text{ is not } S^* \text{-invariant but } S \text{-invariant}\} = \{qH^2: q \text{ is an inner function}\}.$$

Wiener の定理 次の等式がなりたつ。

$$\{E \subset L^2: e^{i\theta}E = E\} = \{\chi_A L^2: A \text{ is a measurable set of } \mathbb{T}\}.$$

Sarason の定理 内部関数 q に対して $N = H^2 \ominus qH^2$ と定める。このとき、次の等式がなりたつ。

$$\{A \in B(N): SA = AS\} = \{T_f: f \in H^\infty, \|A\| = \|f\|_\infty\}.$$

(cf. [2], [5], [7], [8], [14], [27], [35], [36], [41], [42], [44], [50], [51], [52], [56], [61])

2. マジヨライゼーションと因数分解定理

In this section, we use the Helson-Szegő type set $(HS)(r)$ to establish the condition of a, b, c and d satisfying $\|Af\|_2 \geq \|Bf\|_2$ for all f in \mathcal{P} . The main theorem is Theorem 2.4. If $a, b, c, d \in L^\infty$, then this is equivalent to that B is majorized by A on L^2 , i.e., $A^*A \geq B^*B$ on L^2 , i.e., $\|Af\|_2 \geq \|Bf\|_2$ for all f in L^2 .

It is well known that the Helson-Szegő set

$$(HS) = \left\{ e^{u+i\tilde{v}}, u, v \in L^\infty \text{ are real functions, and } \|v\|_\infty < \frac{\pi}{2} \right\}$$

is equal to the set of all Muckenhoupt (A_2) -weights (cf. [5, p.254]). For a function r satisfying $0 \leq r \leq 1$, we define the Helson-Szegő type set $(HS)(r)$ as follows.

$$(HS)(r) = \{ \gamma e^{u+i\tilde{v}}; \gamma \text{ is a positive constant, } u, v \text{ are real functions, } \\ u \in L^1, v \in L^\infty, |v| \leq \pi/2, r^2 e^u + e^{-u} \leq 2 \cos v \}$$

Remark 2.1. If $|v| \leq \pi/2$, then $e^{\tilde{v}} \cos v \in L^1$ (cf. [5, p. 161]), and hence $e^{-\tilde{v}} \cos v \in L^1$. Therefore $(HS)(r) \subset \{W: W > 0, r^2 W \in L^1, W^{-1} \in L^1\}$. If $r^{-1} \in L^\infty$, then $(HS)(r) \subset (HS)$. In [10], we defined the another Helson-Szegő type set which is similar to $(HS)(r)$. If $r > 0$, then W is in $(HS)(r)$ if and only if $W = \gamma e^{u+\tilde{v}}$ where γ is a positive constant, u, v are real functions such that there is a function t satisfying $1 \leq t \leq r^{-1}$, $|v| \leq \cos^{-1}(rt)$ and $|u + \log r| \leq \cosh^{-1}(t)$.

Theorem 2.2. $|\phi| \geq W$ かつ $\int_{\mathbf{T}} (|\phi| - W) dm > 0$ とし

$$r = |\phi|^{-1} \sqrt{|\phi|^2 - W^2}$$

と定める。このとき、以下の条件は互いに同値である。

- (1) There is a k in H^1 such that $|\phi - k| \leq W$.
- (2) There is a non-zero k in H^1 such that $|\phi - k| \leq W$.
- (3) $\log |\phi| \in L^1$ and there are real functions $u \in L^1$ and $v \in L^\infty$, $|v| \leq \pi/2$ such that $\phi e^{-u-\tilde{v}}$ is in $H^{1/2}$ and $r^2 e^u + e^{-u} \leq 2 \cos v$.
- (4) $\log |\phi| \in L^1$ and there is a V in $(HS)(r)$ such that ϕ/V is in $H^{1/2}$.

Theorem 2.3. もし $|\phi|^2 - W_1 W_2 \geq 0$ ならば、以下の条件は互いに同値である。

- (1) For all $(Pf) \in \mathcal{P}_1$ and $(Qf) \in \mathcal{P}_2$,

$$\left| \int_{\mathbf{T}} (Pf) \overline{(Qf)} \phi dm \right| \leq \int_{\mathbf{T}} \frac{|Pf|^2 W_1 + |Qf|^2 W_2}{2} dm.$$

- (2) W_1, W_2 and ϕ satisfy (2.1) or (2.2).

$$(2.1) \quad W_1 \geq 0, W_2 \geq 0, \text{ and } |\phi|^2 - W_1 W_2 = 0.$$

$$(2.2) \quad \log |\phi| \in L^1, W_1 \geq 0, W_2 \geq 0, \text{ and there is a } V \text{ in } (HS)(r) \text{ such that } \phi/V \text{ is in } H^{1/2}, \text{ where}$$

$$r = |\phi|^{-1} \sqrt{|\phi|^2 - W_1 W_2}.$$

In Theorem 2.3, if $|\phi|^2 - W_1 W_2 \leq 0$ then

$$\begin{aligned} \left| \int_{\mathbf{T}} (Pf) \overline{(Qf)} \phi dm \right| &\leq \int_{\mathbf{T}} |(Pf)(Qf)\phi| dm \\ &\leq \int_{\mathbf{T}} |(Pf)(Qf)| \sqrt{W_1 W_2} dm \\ &\leq \int_{\mathbf{T}} \frac{|Pf|^2 W_1 + |Qf|^2 W_2}{2} dm, \end{aligned}$$

and so (1) holds without the condition (2).

Theorem 2.4. 以下の条件は互いに同値である。

(1) For all f in \mathcal{P} ,

$$\int_{\mathbb{T}} |a(Pf) + b(Qf)|^2 dm \geq \int_{\mathbb{T}} |c(Pf) + d(Qf)|^2 dm.$$

(2) a, b, c, d satisfy (2.1) or (2.2).

(2.1) $|a| \geq |c|, |b| \geq |d|$, and $ad - bc = 0$.

(2.2) $\log |a\bar{b} - c\bar{d}| \in L^1, |a| \geq |c|, |b| \geq |d|, \log |a\bar{b} - c\bar{d}| \in L^1$, and there is a V in $(HS)(r)$ such that $(a\bar{b} - c\bar{d})/V$ is in $H^{1/2}$, where $r = |ad - bc|/|a\bar{b} - c\bar{d}|$.

Let W be a non-negative function in L^1 such that $\int_{\mathbb{T}} W dm > 0$. Let $L^2(W)$ be the weighted Lebesgue space with the norm

$$\|f\|_{2,W} = \left\{ \int_{\mathbb{T}} |f|^2 W dm \right\}^{1/2}.$$

Corollary 2.5. もし $A = aP + bQ$ かつ $B = cP + dQ$ ならば, 以下の条件は互いに同値である。

(1) B is majorized by A on $L^2(W)$, i.e., $A^*A \geq B^*B$ on $L^2(W)$, i.e., $\|Af\|_{2,W} \geq \|Bf\|_{2,W}$ for all f in $L^2(W)$.

(2) There is a contraction C on $L^2(W)$ such that B is factorized as $B = CA$.

(3) a, b, c, d satisfy (3.1) or (3.2).

(3.1) $(|a| - |c|)W \geq 0, (|b| - |d|)W \geq 0$, and $(ad - bc)W = 0$.

(3.2) $\log |a\bar{b} - c\bar{d}| W \in L^1, |a| \geq |c|, |b| \geq |d|$, and there is a V in $(HS)(r)$ such that $(a\bar{b} - c\bar{d})W/V$ is in $H^{1/2}$, where $r = |ad - bc|/|a\bar{b} - c\bar{d}|$.

3. 縮小作用素と有界特異積分作用素

In this section, some applications of Theorem 2.4 are given for contractive and bounded singular integral operators on weighted L^2 spaces.

Corollary 3.1. 以下の条件は互いに同値である。

(1) $\alpha P + \beta Q$ is contractive on $L^2(W)$.

(2) α, β and W satisfy (2.1) or (2.2).

(2.1) $\max(|\alpha|, |\beta|)W \leq W$ and $|\alpha - \beta|W = 0$.

(2.2) $\log |1 - \alpha\bar{\beta}| W \in L^1, \max(|\alpha|, |\beta|) \leq 1$, and there is a V in $(HS)(r)$ such that $(1 - \alpha\bar{\beta})W/V$ is in $H^{1/2}$, where $r = |\alpha - \beta|/|1 - \alpha\bar{\beta}|$.

Corollary 3.2. ([10]) 以下の条件は互いに同値である。

- (1) $\alpha P + \beta Q$ is bounded on $L^2(W)$.
- (2) α, β and W satisfy (2.1) or (2.2).
 - (2.1) $|\alpha - \beta|W = 0$.
 - (2.2) There are real functions $u, v \in L^1$, and positive constant ε such that W is in $(HS)(r)$, where $r = \varepsilon|\alpha - \beta|$.
- (3) α, β and W satisfy (3.1) or (3.2).
 - (3.1) $|\alpha - \beta|W = 0$.
 - (3.2) There are real functions $u, v \in L^1$, and a positive constant γ such that $W = e^{u+v}$, $|v| \leq \pi/2$, $|\alpha - \beta|^2 e^u \leq \gamma \cos v$, and $e^{-u} \leq \gamma \cos v$.

Corollary 3.3. ([10]) 以下の条件は互いに同値である。

- (1) For all f in \mathcal{P} ,

$$\int_{\mathbb{T}} |f|^2 W dm \geq \int_{\mathbb{T}} |Pf|^2 U dm.$$

- (2) $\log W \in L^1$, $W \geq U$, and W is in $(HS)(r)$ where $r = \sqrt{U/W}$.

Corollary 3.4. (Koosis) 以下の条件は互いに同値である。

- (1) There is a non-negative function U in L^1 such that $\int_{\mathbb{T}} U dm > 0$ and for all f in \mathcal{P} ,

$$\int_{\mathbb{T}} |f|^2 W dm \geq \int_{\mathbb{T}} |Pf|^2 U dm.$$

- (2) There is a non-negative function U in L^1 such that $\log U \in L^1$ and for all f in \mathcal{P} ,

$$\int_{\mathbb{T}} |f|^2 W dm \geq \int_{\mathbb{T}} |Pf|^2 U dm.$$

- (3) W^{-1} is in L^1 .

4. 拡大作用素と下に有界な特異積分作用素

In this section, some applications of Theorem 2.4 are given for expansive and bounded-below singular integral operators on weighted L^2 spaces.

Corollary 4.1. 以下の条件は互いに同値である。

- (1) $\alpha P + \beta Q$ is expansive on $L^2(W)$.
- (2) α, β and W satisfy (2.1) or (2.2).

(2.1) $\min(|\alpha|, |\beta|)W \geq W$ and $|\alpha - \beta|W = 0$.

(2.2) $\log|\alpha\bar{\beta} - 1|W \in L^1$, $\min(|\alpha|, |\beta|) \geq 1$, and there is a V in $(HS)(r)$ such that $(1 - \alpha\bar{\beta})W/V$ is in $H^{1/2}$, where $r = |\alpha - \beta|/|\alpha\bar{\beta} - 1|$.

Corollary 4.2. 以下の条件は互いに同値である。

(1) $\alpha P + \beta Q$ is bounded-below on $L^2(W)$.

(2) α, β and W satisfy (2.1) or (2.2).

(2.1) There is a positive constant ε such that $\min(|\alpha|, |\beta|)W \geq \varepsilon W$, and $|\alpha - \beta|W = 0$.

(2.2) There is a positive constant ε such that $\log|\alpha\bar{\beta} - \varepsilon|W \in L^1$, $\alpha^{-1}, \beta^{-1} \in L^\infty$, and there is a V in $(HS)(r)$ such that $(\alpha\bar{\beta} - \varepsilon)W/V$ is in $H^{1/2}$, where $r = |\alpha - \beta|\varepsilon/|\alpha\bar{\beta} - \varepsilon|$.

Corollary 4.3. 以下の条件は互いに同値である。ただし, $\int_{\mathbb{T}}|\beta|dm > 0$ とする。

(1) There is a constant $\varepsilon > 0$ such that for all f in \mathcal{P} ,

$$\int_{\mathbb{T}}|(\alpha P + \beta Q)f|^2 W dm \geq \varepsilon \int_{\mathbb{T}}|Pf|^2 W dm.$$

(2) $\log|\alpha\beta|W \in L^1$, $|a| \geq |c|$ and there is a V in $(HS)(r)$ such that $a\bar{b}/V$ is in $H^{1/2}$, where $r = |c/a|$.

Corollary 4.3 A. 以下の条件は互いに同値である。ただし, $\int_{\mathbb{T}}|bc|dm > 0$ とする。

$$\int_{\mathbb{T}}|(aP + bQ)f|^2 dm \geq \int_{\mathbb{T}}|cPf|^2 dm.$$

(2) $\log|ab| \in L^1$, $|a| \geq |c|$ and there is a V in $(HS)(r)$ such that $a\bar{b}/V$ is in $H^{1/2}$, where $r = |c/a|$.

Corollary 4.4. 以下の条件は互いに同値である。ただし, $\int_{\mathbb{T}}|\alpha - \beta|dm > 0$ とする。

(1) $\alpha P + \beta Q$ is contractive on L^2 .

(2) $\log|1 - \alpha\bar{\beta}|dm \in L^1$, $\max(|\alpha|, |\beta|) \leq 1$, and there is a V in $(HS)(r)$ such that $(1 - \alpha\bar{\beta})/V$ is in $H^{1/2}$, where $r = |\alpha - \beta|/|1 - \alpha\bar{\beta}|$.

Corollary 4.5. 以下の条件は互いに同値である。ただし, $\int_{\mathbb{T}}|a - b|W dm > 0$ とする。

(1) For all f in \mathcal{P} ,

$$\int_{\mathbb{T}}|f|^2 W dm \geq \int_{\mathbb{T}}|(aP + bQ)f|^2 U dm.$$

(2) $\log|W - a\bar{b}U| \in L^1$, $\max(|a|^2, |b|^2)U \leq W$, and there is a V in $(HS)(r)$ such that $(W - a\bar{b}U)/V$ is in $H^{1/2}$, where $r = |a - b|\sqrt{WU}/|W - a\bar{b}U|$.

Theorem 4.6. 以下の条件は互いに同値である。ただし, $\int_{\mathbb{T}}|bc|dm > 0$ とする。

(1) For all f in \mathcal{P} ,

$$\int_{\mathbb{T}}|(aP+bQ)f|^2 W dm \geq \int_{\mathbb{T}}|(cP+dQ)f|^2 U dm.$$

(2) $|a|^2 W \geq |c|^2 U$, $|b|^2 W \geq |d|^2 U$, and there is a positive function V in $(HS)(r)$ such that $(a\bar{b}W - c\bar{d}U)/V$ is a non-negative function in $H^{1/2}$, where $r^2 = |ad - bc|^2 WU / |a\bar{b}W - c\bar{d}U|^2$.

Corollary 4.7. 以下の条件は互いに同値である。ただし, $\int_{\mathbb{T}}|\alpha|dm > 0, |\alpha| \leq 1$ とする。

(1) For all f in \mathcal{P} ,

$$\int_{\mathbb{T}}|\alpha P f|^2 W dm \leq \int_{\mathbb{T}}|f|^2 W dm.$$

(2) There are real functions $u, v \in L^1$, and a positive constant γ such that $W = \gamma e^{u+\tilde{v}}, |v| \leq \pi/2$, and

$$|\alpha|^2 e^u + e^{-u} \leq 2 \cos v.$$

Theorem 4.8. 以下の条件は互いに同値である。ただし, $\int_{\mathbb{T}}|a-b|dm > 0$ とする。

(1) For all f in \mathcal{P} ,

$$\int_{\mathbb{T}}|(aP+bQ)f|^2 dm \leq \int_{\mathbb{T}}|f|^2 dm.$$

(2) $\max(|a|, |b|) \leq 1$, and there is a positive function V in $(HS)(r)$ such that $(1 - a\bar{b})/V$ is a non-negative function in $H^{1/2}$, where $r = |a-b|\sqrt{WU} / |W - a\bar{b}U|$.

Corollary 4.9. 以下の条件は互いに同値である。ただし, $\int_{\mathbb{T}}|ad-bc|dm > 0, |a| \geq |c|, |b| \geq |d|, a\bar{b} - c\bar{d} \geq 0$ とする。

(1) For all f in \mathcal{P} ,

$$\int_{\mathbb{T}}|(aP+bQ)f|^2 dm \leq \int_{\mathbb{T}}|f|^2 dm.$$

(2) $\max(|a|, |b|) \leq 1$, and $1 - a\bar{b}$ is in $(HS)(r)$, where $r^2 = |a-b|^2 / |1 - a\bar{b}|^2$.

Theorem A. (Cotlar and Sadosky) 以下の条件は互いに同値である。

(1) For all $(Pf) \in \mathcal{P}_1$ and $(Qf) \in \mathcal{P}_2$,

$$\int_{\mathbb{T}}\{|Pf|^2 W_1 + |Qf|^2 W_2 + 2Re((Pf)\overline{(Qf)\phi})\} dm \geq 0.$$

(2) $W_1 \geq 0, W_2 \geq 0$, and there is an $k \in H^1$ such that $|\phi - k|^2 \leq W_1 W_2$.

Theorem 4.10. 以下の条件は互いに同値である。ただし、 $\phi^2 - W_1W_2 \geq 0$ and $\int_{\mathbb{T}}(\phi^2 - W_1W_2)dm > 0$ とする。

(1) For all $(Pf) \in \mathcal{P}_1$ and $(Qf) \in \mathcal{P}_2$,

$$\int_{\mathbb{T}}\{|Pf|^2W_1 + |Qf|^2W_2 + 2Re((Pf)\overline{(Qf)}\phi)\}dm \geq 0.$$

(2) $W_1 \geq 0, W_2 \geq 0$, and there is a positive function V in $(HS)(r)$ and non-negative constant γ such that $\phi = \gamma V$, where $r = (|\phi|^2 - W_1W_2)/|\phi|^2$.

Let m be a normalized Lebesgue measure $\frac{d\theta}{2\pi}$ on the unit circle \mathbb{T} . Let $a, b \in L^\infty = L^\infty(m)$ satisfy

$|a - b| > 0$. Let $w \in L^1 = L^1(m)$ satisfy $w > 0$. The Riesz projection P and $Q = I - P$ can be written as

$$P = \frac{I + S}{2}, Q = \frac{I - S}{2}, \text{ where}$$

$$(Sf)(\zeta) = \frac{1}{\pi i} \int_{\mathbb{T}} \frac{f(z)}{z - \zeta} dz,$$

in the sense of Cauchy's principal value integral, where $\zeta \in \mathbb{T}$, and $f \in L^1$. Hence $P + Q = I$ and $P - Q = S$. Let \bar{f} denote the harmonic conjugate function of f . Then $\bar{f} = -i(Sf - \hat{f}(0))$, where $\hat{f}(0)$ is the 0-th Fourier coefficient of f .

For $1 < p < \infty$, $L^p(w)$ -norm is:

$$\|f\|_{p,w} = \|f\|_{L^p(w)} = \left(\int_{\mathbb{T}} |f|^p w dm \right)^{1/p}$$

If $p = 2$, then we write $\|f\|_w = \|f\|_{2,w}$.

Helson-Szegő theorem (1960)

$\|Pf\|_w \leq c\|f\|_w$, (for all f), for some positive constant c .

$\Leftrightarrow \text{dist}\left(\frac{\bar{h}}{h}, H^\infty\right) < 1$, where h is an outer function satisfying $w = |h|^2$ a.e.

\Leftrightarrow There are real functions $u, v \in L^\infty, |v| < \pi/2$ such that $w = \exp(u + iv)$ a.e.

Widom-Devinatz theorem (1960)

Let $|a| = 1$ a.e. Then

(1) $\|(aP + Q)f\|_w \geq \varepsilon\|f\|_w$, (for all f), for some positive constant ε .

$$\Leftrightarrow \text{dist}(a, H^\infty) < 1.$$

(2) $aP+Q$ has a bounded inverse operator on L^2 .

\Leftrightarrow There is an outer function k such that $\|a-k\|_\infty < 1$.

\Leftrightarrow There are real functions $u, v \in L^\infty, |v| < \pi/2$ and a complex constant $\gamma, |\gamma|=1$ such that $a = \gamma \exp(i(\bar{u} + v))$ a.e.

Hunt-Muckenhoupt-Wheeden theorem (1973)

Let $1 < p < \infty$. Then

$\|Pf\|_{p,w} \leq c\|f\|_{p,w}$ (for all f), for some positive constant c .

$$\Leftrightarrow \sup_I \left(\frac{1}{m(I)} \int_I w dm \right) \left(\frac{1}{m(I)} \int_I w^{-1/(p-1)} dm \right)^{p-1} < \infty$$

Rochberg theorem (1977)

Let $1 < p < \infty, a \in L^\infty, w \in (A_p)$. Then

$aP+Q$ has a bounded inverse operator on $L^p(w)$.

\Leftrightarrow There are real functions $U \in L^\infty, V \in L^1$ and a complex constant $\gamma, |\gamma|=1$ such that

$$w \exp\left(\frac{p}{2}V\right) \in (A_p), a = \gamma \exp(U - i\bar{V}) \text{ a.e.}$$

Cotlar-Sadosky Theorem (1979)

Let c be a constant satisfying $c \geq 1$. Then

$\|Sf\|_w \leq c\|f\|_w$ (for all f).

\Leftrightarrow There is a $k \in H^1$ such that $|w-k|^2 \leq \left(1 - \left(\frac{2c}{c^2+1}\right)^2\right)w^2$ a.e.

\Leftrightarrow There are real functions $u, v \in L^\infty$ such that $w = \exp(u + \bar{v} + \text{const.})$,

$$|v| \leq \frac{\pi}{2} - \arcsin\left(\frac{2c}{c^2+1}\right), |u| \leq \cosh^{-1}\left(\frac{c^2+1}{2c} \cos v\right) \text{ a.e. where}$$

$$\cosh^{-1} x = \log(x + \sqrt{x^2 - 1}).$$

Nakazi theorem (1993)

$1 < p < \infty, a \in L^\infty, w \in (A_p)$ とする。

(1) Let h be an outer function satisfying $w=|h|^p$ a.e. Then

$\|(aP+Q)f\|_{p,w} \geq \varepsilon \|f\|_{p,w}$ (for all f), for some positive constant ε .

\Leftrightarrow There are functions k and a_0 such that $a=k(h/\bar{h})a_0$ a.e. where $k, k^{-1} \in H^\infty$, $|a_0|=1$ a.e., and

$\|(a_0P+Q)f\|_p \geq \varepsilon \|f\|_p$ (for all f), for some positive constant ε .

(2) There is an inner function q such that $\bar{q}aP+Q$ has a bounded inverse operator on $L^p(w)$.

$\Rightarrow \|(aP+Q)f\|_{p,w} \geq \varepsilon \|f\|_{p,w}$ (for all f), for some positive constant ε .

If $p=2$, then the converse is also true.

Let $a, b \in L^\infty$, and let c be a positive constant. Then we consider the positive function w in L^1 satisfying $\|(aP+bQ)f\|_w \leq c \|f\|_w$ (for all f).

Let $|a\bar{b}-c^2| > 0$ a.e. and let

$$d_c = d_c(a, b) = \frac{|(a-b)c|}{|a\bar{b}-c^2|}$$

Since

$$\|aP+bQ\|_{B(L^2(w))} \leq c \Rightarrow \max(|a|, |b|) \leq c \text{ a.e.},$$

we have $|d_c(a, b)| \leq 1$ a.e. If $\max(|a|, |b|) \leq c$ a.e., then

$$\|aP+bQ\|_{B(L^2(w))} \leq c \Leftrightarrow \text{There is a } k \in H^1 \text{ such that } \left| 1 - \frac{k}{(a\bar{b}-c^2)w} \right|^2 \leq 1 - d_c^2 \text{ a.e.}$$

If a, b are complex constants, then

$$c = \|aP+bQ\|_{B(L^2(w))} \Rightarrow d_c(a, b) = \frac{1}{\|P\|_{B(L^2(w))}}$$

Hence we have the Cotlar-Sadosky's result:

$$c = \|S\|_{B(L^2(w))} \Rightarrow c = \|P-Q\|_{B(L^2(w))} \Rightarrow d_c(1, -1) = \frac{2c}{c^2+1} = \frac{1}{\|P\|_{B(L^2(w))}}.$$

Theorem 4.11. 以下の条件は互いに同値である。

(1) $\|(aP+bQ)f\|_w \leq c \|f\|_w$ (for all f).

(2) There is an inner function q and real functions $t, u \in L^1, v \in L^\infty$ such that

$$w = \frac{1}{|a\bar{b}-c^2|} \exp(u + \bar{v} + t) \text{ a.e.}, \frac{a\bar{b}-c^2}{|a\bar{b}-c^2|} = q \exp(i\bar{t}) \text{ a.e.},$$

$$|v| \leq \frac{\pi}{2} - \arcsin d_c \text{ a.e.}, d_c^2 e^u + e^{-u} \leq 2 \cos v \text{ a.e.}$$

(3) There is an inner function q and real functions $t \in L^1, v \in L^\infty, u_0$ such that

$$w = \frac{1}{|a-b|} \exp(u_0 + \bar{v} + t) \text{ a.e.}, \quad \frac{\bar{a}\bar{b} - c^2}{|\bar{a}\bar{b} - c^2|} = q \exp(it) \text{ a.e.},$$

$$|v| \leq \frac{\pi}{2} - \arcsin d_c \text{ a.e.}, \quad |u_0| \leq \cosh^{-1} \left(\frac{\cos v}{d_c} \right) \text{ a.e.}$$

Theorem 4.12. 以下の条件は互いに同値である。ただし $|a-b| > 0$ a.e., $\max(|a|, |b|) \leq c$ a.e., $|\bar{a}\bar{b} - c^2| > 0$ a.e. とする。

$$\exp(-s) |\bar{a}\bar{b} - c^2| w \in L^1, \quad \gamma \exp(is) = \frac{\bar{a}\bar{b} - c^2}{|\bar{a}\bar{b} - c^2|} \text{ a.e.}$$

Then (1) ~ (3) are equivalent.

(1) $\|(aP + bQ)f\|_w \leq c \|f\|_w$, (for all f).

(2) There are real functions $u \in L^1, v \in L^\infty$ such that

$$w = \frac{1}{|\bar{a}\bar{b} - c^2|} \exp(u + \bar{v} + s + \text{const.}) \text{ a.e.},$$

$$|v| \leq \frac{\pi}{2} - \arcsin d_c \text{ a.e.}, \quad d_c^2 e^u + e^{-u} \leq 2 \cos v \text{ a.e.}$$

(3) There are real functions $v \in L^\infty, u$ such that

$$w = \frac{1}{|a-b|} \exp(u_0 + \bar{v} + s + \text{const.}) \text{ a.e.},$$

$$|v| \leq \frac{\pi}{2} - \arcsin d_c \text{ a.e.}, \quad |u_0| \leq \cosh^{-1} \left(\frac{\cos v}{d_c} \right) \text{ a.e.}$$

Theorem 4.13. 以下の条件は互いに同値である。ただし $|a-b| > 0$ a.e., $\max(|a|, |b|) \leq c$ a.e., $\bar{a}\bar{b} \in H^\infty, \bar{a}\bar{b} \neq c^2$ とする。

(1) $\|(aP + bQ)f\|_w \leq c \|f\|_w$, (for all f).

(2) There are real functions $u \in L^1, v \in L^\infty$ such that

$$w = \exp(u + \bar{v} + \text{const.}) \text{ a.e.},$$

$$|v| \leq \frac{\pi}{2} - \arcsin d_c \text{ a.e.}, \quad d_c^2 e^u + e^{-u} \leq 2 \cos v \text{ a.e.}$$

(3) There are real functions $v \in L^\infty, u_0$ such that

$$w = d_c^{-1} \exp(u_0 + \bar{v} + \text{const.}) \text{ a.e.},$$

$$|v| \leq \frac{\pi}{2} - \arcsin d_c \text{ a.e.}, \quad |u_0| \leq \cosh^{-1} \left(\frac{\cos v}{d_c} \right) \text{ a.e.}$$

By Theorem 4.14, we generalize the next result. Let $c > 1$, and let $w(z) = \frac{1}{|z - c^2|^2}$. Then $(c^2 - \bar{z})w = \frac{1}{c^2 - z} \in H^1$. Hence,

$$\begin{aligned} \|(\bar{z}P + Q)f\|_w^2 &= c^2\|f\|_w^2 - (c^2 - 1)(\|Pf\|_w^2 + \|Qf\|_w^2) \\ &\leq c^2\|f\|_w^2, \text{ (for all } f). \end{aligned}$$

Theorem 4.14. 以下の条件は互いに同値である。ただし $|a - b| > 0$ a.e., $\max(|a|, |b|) \leq c$ a.e., $\bar{a}b \in H^\infty$, $\bar{a}b \neq c^2$ とする。

- (1) $\|(aP + bQ)f\|_w \leq c\|f\|_w$, (for all f).
- (2) There are real functions $u \in L^1, v \in L^\infty$ such that

$$w = \frac{1}{|\bar{a}b - c^2|^2} \exp(u + \bar{v} + \text{const.}) \text{ a.e.},$$

$$|v| \leq \frac{\pi}{2} - \arcsin d_c \text{ a.e.}, \quad d_c^2 e^u + e^{-u} \leq 2 \cos v \text{ a.e.}$$

- (3) There are real functions $v \in L^\infty, u_0$ such that

$$w = \frac{1}{|a - b| \cdot |\bar{a}b - c^2|} \exp(u_0 + \bar{v} + \text{const.}) \text{ a.e.},$$

$$|v| \leq \frac{\pi}{2} - \arcsin d_c \text{ a.e.}, \quad |u_0| \leq \cosh^{-1} \left(\frac{\cos v}{d_c} \right) \text{ a.e.}$$

Let ε be a positive constant, and let $a, b \in L^\infty$. Then we consider the positive function w in L^1 satisfying

$$\|(aP + bQ)f\|_w \geq \varepsilon\|f\|_w, \text{ (for all } f).$$

Let $|\bar{a}b - \varepsilon^2| > 0$ a.e., and let

$$d_\varepsilon(a, b) = \frac{|(a - b)\varepsilon|}{|\bar{a}b - \varepsilon^2|}.$$

Since

$$\|(aP + bQ)f\|_w \geq \varepsilon\|f\|_w, \text{ (for all } f) \Rightarrow \min(|a|, |b|) \geq \varepsilon \text{ a.e.},$$

it follows that $|d_\varepsilon(a, b)| \leq 1$ a.e. Hence, if $\min(|a|, |b|) \geq \varepsilon$ a.e., then

$$\|(aP+bQ)f\|_w \geq \varepsilon \|f\|_w \text{ (for all } f)$$

⇔ There is a $k \in H^1$ such that

$$\left| 1 - \frac{k}{(\bar{a}b - \varepsilon^2)w} \right|^2 \leq 1 - d_\varepsilon^2 \text{ a.e. (cf. [8], [9], [13]).}$$

Theorem 4.15. 以下の条件は互いに同値である。ただし $|a-b| > 0$ a.e., $\min(|a|, |b|) \geq \varepsilon$ a.e., $|\bar{a}b - \varepsilon^2| > 0$ a.e. とする。

(1) $\|(aP+bQ)f\|_w \geq \varepsilon \|f\|_w$ (for all f).

(2) There is an inner function q and real functions $t, u \in L^1, v \in L^\infty$ such that

$$w = \frac{1}{|\bar{a}b - \varepsilon^2|} \exp(u + \bar{v} + t) \text{ a.e., } \frac{\bar{a}b - \varepsilon^2}{|\bar{a}b - \varepsilon^2|} = q \exp(i\bar{t}) \text{ a.e.,}$$

$$|v| \leq \frac{\pi}{2} - \arcsin d_\varepsilon \text{ a.e., } d_\varepsilon^2 e^u + e^{-u} \leq 2 \cos v \text{ a.e.}$$

(3) There is an inner function q and real functions $t \in L^1, v \in L^\infty, u_0$ such that

$$w = \frac{1}{|a-b|} \exp(u_0 + \bar{v} + t) \text{ a.e., } \frac{\bar{a}b - \varepsilon^2}{|\bar{a}b - \varepsilon^2|} = q \exp(i\bar{t}) \text{ a.e.,}$$

$$|v| \leq \frac{\pi}{2} - \arcsin d_\varepsilon \text{ a.e., } |u_0| \leq \cosh^{-1} \left(\frac{\cos v}{d_\varepsilon} \right) \text{ a.e.}$$

Theorem 4.16 以下の条件は互いに同値である。ただし $|a-b| > 0$ a.e., $\min(|a|, |b|) \geq \varepsilon$ a.e., $|\bar{a}b - \varepsilon^2| > 0$ a.e. とし $s \in L^1$ と complex constant $\gamma, |\gamma| = 1$ が存在して

$$\exp(-s) |\bar{a}b - \varepsilon^2| w \in L^1, \quad \gamma \exp(is) = \frac{\bar{a}b - \varepsilon^2}{|\bar{a}b - \varepsilon^2|} \text{ a.e.}$$

を満足していることを仮定する。

(1) $\|(aP+bQ)f\|_w \geq \varepsilon \|f\|_w$ (for all f).

(2) There are real functions $u \in L^1, v \in L^\infty$ such that

$$w = \frac{1}{|\bar{a}b - \varepsilon^2|} \exp(u + \bar{v} + s + \text{const.}) \text{ a.e.,}$$

$$|v| \leq \frac{\pi}{2} - \arcsin d_\varepsilon \text{ a.e., } d_\varepsilon^2 e^u + e^{-u} \leq 2 \cos v \text{ a.e.}$$

(3) There are real functions $v \in L^\infty, u$ such that

$$w = \frac{1}{|a-b|} \exp(u_0 + \bar{v} + s + \text{const.}) \text{ a.e.,}$$

$$|v| \leq \frac{\pi}{2} - \arcsin d_\varepsilon \text{ a.e.}, \quad |u| \leq \cosh^{-1} \left(\frac{\cos v}{d_\varepsilon} \right) \text{ a.e.}$$

By Theorem 4.17, we generalize the next result. Let $0 < \varepsilon \leq 1$, and let

$$w(z) = \left| \frac{z-1}{z-\varepsilon^2} \right|^2.$$

Since

$$(z-\varepsilon^2)w = \frac{(z-1)(\bar{z}-1)}{\bar{z}-\varepsilon^2} = \frac{-(1-z)^2}{1-\varepsilon^2 z} \in H^1,$$

it follows that

$$\|(zP+Q)f\|_w^2 = \varepsilon^2 \|f\|_w^2 + (1-\varepsilon^2)(\|Pf\|_w^2 + \|Qf\|_w^2) \geq \varepsilon^2 \|f\|_w^2 \text{ (for all } f\text{)}.$$

Theorem 4.17. $0 < \varepsilon < 1$, $a, c \in L^\infty$, $|a-b| > 0$ a.e., $\min(|a|, |b|) \geq \varepsilon$ a.e. とし $a\bar{b}$ は non-constant inner function satisfying

$$\left| \frac{a\bar{b}-\varepsilon^2}{a\bar{b}-1} \right|^2 w \in L^1$$

とする。このとき $aP+bQ$ は unbounded operator であり、以下の条件は互いに同値である。

- (1) $\|(aP+bQ)f\|_w \geq \varepsilon \|f\|_w$ (for all f).
- (2) There are real functions $u \in L^1, v \in L^\infty$ such that

$$w = \left| \frac{a\bar{b}-1}{a\bar{b}-\varepsilon^2} \right|^2 \exp(u + \bar{v} + \text{const.}) \text{ a.e.},$$

$$|v| \leq \frac{\pi}{2} - \arcsin d_\varepsilon \text{ a.e.}, \quad d_\varepsilon^2 e^u + e^{-u} \leq 2 \cos v \text{ a.e.}$$

- (3) There are real functions $v \in L^\infty, u_0$ such that

$$w = \frac{|a\bar{b}-1|^2}{|a-b| \cdot |a\bar{b}-\varepsilon^2|} \exp(u_0 + \bar{v} + \text{const.}) \text{ a.e.},$$

$$|v| \leq \frac{\pi}{2} - \arcsin d_\varepsilon \text{ a.e.}, \quad |u_0| \leq \cosh^{-1} \left(\frac{\cos v}{d_\varepsilon} \right) \text{ a.e.}$$

Theorem 4.18. もし $|a-b| > 0$ a.e. であり、 $a\bar{b}$ が non-constant inner function であるならば $aP+bQ$ は not bounded-below in $L^2(w)$ である。

Theorem 4.17 has the following equivalent formulation. For $a, b, c, d \in L^\infty$, the condition (1) of Theorem 4.18 implies that B is majorized by A on L^2 , i.e., $A^*A \geq B^*B$ on $L^2(W)$, where

$$A = aP + bQ \text{ and } B = cP + dQ.$$

Corollary 4.19. $W \in (HS)$ とし α, β は complex functions in L^∞ であり $|\alpha| = |\beta|$ を満足しているとする。このとき、以下の条件は互いに同値である。

- (1) $\alpha P + \beta Q$ is bounded-below on $L^2(W)$.
- (2) $\alpha^{-1} \in L^\infty$, and there is a V in (HS) such that $\alpha W/V$ is in $H^{1/2}$.

Corollary 4.20. 以下の条件は互いに同値である。

- (1) There is a non-negative function U in L^1 such that for all f in \mathcal{P} ,

$$\int_{\mathbb{T}} |f|^2 U dm \geq \int_{\mathbb{T}} |Pf|^2 W dm.$$

- (2) There is a positive constant γ and real function v satisfying $|v| < \pi/2$ and $W \leq \gamma e^{\tilde{v}} \cos v$.

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