タイトル	Exact Solutions of a Difference-Difference Nonlinear Equation with the mKdV Type Soliton
著者	YOSHIDA, Fumio; SETO, Kenji
引用	工学研究 : 北海学園大学大学院工学研究科紀要, 8: 33-40
発行日	2008-09-28

研究論文

Exact Solutions of a Difference-Difference Nonlinear Equation with the mKdV Type Soliton

Fumio YOSHIDA* and Kenji SETO**

Abstract

We propose a difference-difference nonlinear equation $\Delta_j^c \mathrm{Tan}^{-1}(\phi_{n,j}) = \Delta_n^c \mathrm{Tan}^{-1}(\gamma\phi_{n,j})$, and present some exact solutions, namely from the 1-soliton solution up to the 4-soliton one. Furthermore we develop an argument for the N-soliton solution, and discuss the possibility that the equation belongs to an integrable system.

Keywords: difference-difference nonlinear equation, mKdV type soliton, integrability

1. Introduction

Recently, the nonlinear equations having discrete space-time variables, namely the discretized soliton equations are researched intensively from the viewpoint of the integrable equations.[1)] One standard criterion that a certain soliton equation belongs to the integrable system indicates that the equation has the N-soliton solution. As an effective means to discretize a nonlinear equation keeping this integrability, Hirota's bilinear method is well known.[2)] And the discrete time Toda lattice equation or discrete KdV one has been found later.[3−5)]

If one of the model equations that has the soliton solution can be found, it is very effective for the research of the nonlinear system. For this reason, we have looked for a discretized nonlinear equation which has the soliton solution, and found one model equation.

Here we propose a discrete space-time nonlinear difference-difference equation

$$\Delta_j^c \mathrm{Tan}^{-1}(\phi_{n,j}) = \Delta_n^c \mathrm{Tan}^{-1}(\gamma\phi_{n,j}), \tag{1}$$

in which n and j are integers describing the lattice points of space and time respectively, and 1st order central difference operator Δ_j^c is defined as $\Delta_j^c f_j = (f_{j+1} - f_{j-1})/2$, and the same is applied to the suffix n. The parameter γ is a real constant corresponding to the time scale, and relating to continuous time t as $t = \gamma j$. As is known from the linear approximation of the equation, the positive /negative sign of γ corresponds to the left/right-traveling wave respectively. If $\gamma = \pm 1$, the equation reduces to linear one. So, we assume that γ is not equal to ± 1 . If $|\gamma| > 1$, redefining $\gamma\phi_{n,j}$ as $\phi_{n,j}$, and interchanging n and j, we get the equation where γ is substituted for $1/\gamma$. So, we assume that $0 < |\gamma| < 1$, without loss of generality.

In this paper, we offer some exact solutions of the equation, namely 1-soliton, 2-soliton, optical soliton, complex soliton, 3-, 4-soliton and N-soliton, and discuss the possibility that the equation belongs to an integrable system.

* Graduate School of Engineering (Civil Eng.), Hokkai-Gakuen University
** Department of Architecture and Building Engineering, Hokkai-Gakuen University

2. Transformation of Equation (1)

In the following transformation from the variable $\phi_{n,j}$ to $f_{n,j}$,

$$\mathrm{Tan}^{-1}(\gamma\phi_{n,j}) = 2\Delta_j^c \mathrm{Tan}^{-1}(f_{n,j}), \tag{2}$$

the equation (1) is transformed to

$$\frac{\Delta_j^c f_{n,j}}{1+f_{n,j+1}f_{n,j-1}} = \gamma \frac{\Delta_n^c f_{n,j}}{1+f_{n+1,j}f_{n-1,j}} \left(= \frac{\gamma}{2}\phi_{n,j} \right). \tag{3}$$

It is noticed that the equation (3) is invariant under a linear transformation T_θ for an arbitrary θ,

$$g_{n,j} = T_\theta(f_{n,j}) \equiv \frac{\cos\theta \cdot f_{n,j} - \sin\theta}{\sin\theta \cdot f_{n,j} + \cos\theta}. \tag{4}$$

Therefore, when a solution $f_{n,j}$ is given, $g_{n,j}$ also satisfies the equation. The transformation (4) has following properties

$$T_{\theta_1}T_{\theta_2} = T_{\theta_1+\theta_2}, \quad T_0 = 1, \quad T_\theta^{-1} = T_{-\theta}. \tag{5}$$

According to this, the transformation T_θ forms a rotation group.

3. 1-Soliton Solution

In the equation (3), if we put $f_{n,j} = Z \equiv \exp(kn + \omega j + \delta)$, we get the dispersion relation

$$\sinh\omega = \gamma \sinh k, \tag{6}$$

and the mKdV type 1-soliton solution

$$\phi_{n,j} = \sinh k \ \mathrm{sech}(kn + \omega j + \delta), \tag{7}$$

where k, ω, δ are constants corresponding to wave number, angular frequency and initial phase. This solution describes the left/right-traveling wave for positive/negative γ respectively.

Assume long wave continuous limit, $\gamma \to 0$, $k \to 0$, $t = \gamma j$, $x = n$, and the equation (1) reduces to the mKdV one

$$\frac{\partial\phi}{\partial t} - \left(1+\phi^2\right)\frac{\partial\phi}{\partial x} - \frac{1}{6}\frac{\partial^3\phi}{\partial x^3} = 0, \tag{8}$$

and the solution (7) also reduces to the mKdV soliton solution.

4. Decomposition to Bilinear Form

We introduce the fractional transformation $f_{n,j} = F_{n,j}/G_{n,j}$, in order to decompose the equation (3). The result is

$$\frac{\sinh(D_j)F\cdot G}{\cosh(D_j)(F\cdot F + G\cdot G)} = \gamma \frac{\sinh(D_n)F\cdot G}{\cosh(D_n)(F\cdot F + G\cdot G)}, \tag{9}$$

where D_j is Hirota's D operator defined by

$$\sinh(D_j)F\cdot G = F\sinh(\overleftarrow{\partial_j} - \overrightarrow{\partial_j})G. \tag{10}$$

The equation (9) can be decomposed to two bilinear coupled equations

$$\begin{aligned}\sinh(D_j)F\cdot G=\gamma\sinh(D_n)F\cdot G,\\ \cosh(D_j)(F\cdot F+G\cdot G)=\cosh(D_n)(F\cdot F+G\cdot G).\end{aligned}\tag{11}$$

If we put $F=Z=\exp(kn+\omega j+\delta),\ G=1,$ we can get the 1-soliton solution (6), (7).

5. 2-Soliton Solution

To obtain the 2-soliton solution, we assume in (11),

$$F=Z_1+Z_2,\quad G=1+A_{12}Z_1Z_2,\tag{12}$$

where $Z_i=\exp(k_in+\omega_ij+\delta_i),\ (i=1,2),\ A_{12}$ is a coupling constant of two solitons. Substituting (12) in (11), we obtain the dispersion relation for each soliton,

$$\sinh\omega_i=\gamma\sinh k_i,\quad (\ i=1,\ 2\ ),\tag{13}$$

and the coupling constant

$$A_{12}=-\frac{\cosh(\omega_1-\omega_2)-\cosh(k_1-k_2)}{\cosh(\omega_1+\omega_2)-\cosh(k_1+k_2)}.\tag{14}$$

This expression gives the phase shift for the collision of two solitons. This is similar to that of Toda lattice or mKdV equations.

Here, we compare the expression (14) with the coupling constant of the discrete time Toda lattice equation

$$\Delta_j^2\ln(1+\phi^2)=\Delta_n^2\ln(1+\gamma^2\phi^2),\tag{15}$$

where Δ_j^2 is 2nd order difference operator which operates as $\Delta_j^2F_j=F_{j+1}-2F_j+F_{j-1}$. In this case, the coupling constant A_{Toda} is obtained as

$$A_{\mathrm{Toda}}=-\frac{\sinh^2(\omega_1-\omega_2)-\gamma^2\sinh^2(k_1-k_2)}{\sinh^2(\omega_1+\omega_2)-\gamma^2\sinh^2(k_1+k_2)},\tag{16}$$

under the restriction of the dispersion relation $\sinh\omega_i=\pm\gamma\sinh k_i,\ (\ i=1,\ 2\ )$. This is slightly different from (14). But if we restrict the case where two solitons go to the same direction, the numerator and denominator of (16) are factorized, and (16) coincides with (14) when the sign is changed. On the other hand, the case where two solitons go to the opposite direction, both expressions can not be compared, because the solitons are restricted to move one side only in our case.

6. Optical Soliton

In the 1-soliton solution (6) (7), if we adopt a complex wave number $k=k_{\mathrm{R}}+\pi\,\mathrm{i},$ we get the 1-soliton solution

$$\phi_{n,j}=(-1)^{n+1}\sinh k_{\mathrm{R}}\,\mathrm{sech}(k_{\mathrm{R}}n+\omega j+\delta),\tag{17}$$

and the dispersion relation

$$\sinh\omega=-\gamma\sinh k_{\mathrm{R}}.\tag{18}$$

This solution $\phi_{n,j}$ changes the sign alternately for each n point, so we call this type of soliton as an optical one. It is also noticed that this type of soliton propagates to the counter direction with the ordinary one despite the definite parameter γ.

7. Complex Soliton

In the 2-soliton solution (12) (13), we assume that the two wave numbers k_1, k_2, angular frequencies ω_1, ω_2 and initial phases δ_1, δ_2 form each complex conjugate respectively, namely, $k_1 = \overline{k_2} = k + \mathrm{i}\,k'$, $\omega_1 = \overline{\omega_2} = \omega + \mathrm{i}\,\omega'$, $\delta_1 = \overline{\delta_2} = \delta + \mathrm{i}\,\delta'$. By decomposing the dispersion relation $\sinh(\omega + \mathrm{i}\,\omega') = \gamma \sinh(k + \mathrm{i}\,k')$ into the real and imaginary part, we obtain the coupled equations,

$$\begin{cases} \sinh\omega\cos\omega' = \gamma\sinh k\cos k' \\ \cosh\omega\sin\omega' = \gamma\cosh k\sin k'. \end{cases} \tag{19}$$

These real part variables $k,\ \omega,\ \delta$ define the envelope of the soliton, and the imaginary part ones $k',\ \omega',\ \delta'$ define the ripple.

From the equations (19), we can get the angular frequencies $\omega,\ \omega'$ as the functions of the wave numbers $k,\ k'$,

$$\begin{cases} \cosh\omega = \left(\sqrt{A+B} + \sqrt{A-B}\right)/2 \\ \sin\omega' = \left(\sqrt{A+B} - \sqrt{A-B}\right)/2, \end{cases} \tag{20}$$

in which A and B are defined as

$$A = 1 + \gamma^2(\sinh^2 k + \sin^2 k'), \quad B = 2\gamma\cosh k\sin k'. \tag{21}$$

If we consider a special case of $k' = \pi/2$, we get

$$\begin{cases} \cosh\omega = \gamma\cosh k,\ \sin\omega' = 1, & \text{for } \gamma\cosh k > 1, \\ \cosh\omega = 1,\ \ \sin\omega' = \gamma\cosh k, & \text{for } \gamma\cosh k < 1, \end{cases} \tag{22}$$

and if the case of $k' = \pi$,

$$\cosh\omega = \sqrt{1 + \gamma^2\sinh^2 k}, \qquad \sin\omega' = 0. \tag{23}$$

The second expression of (22) shows $\omega = 0$, so the envelope of the soliton becomes stationary. On the other hand, from (23) the solution satisfying $k' = \pi,\ \omega' = 0$, can be existent, which means that the ripple becomes stationary, and the soliton in this case is nothing but the optical one.

The soliton solution in general in this section is obtained from (12) as,

$$\begin{aligned} F &= 2\cos(k'n + \omega'j + \delta')\exp(kn + \omega j + \delta) \\ G &= 1 + A_{12}\exp[2(kn + \omega j + \delta)], \end{aligned} \tag{24}$$

where the coupling constant A_{12} becomes

$$A_{12} = -\frac{\cos(2\omega') - \cos(2k')}{\cosh(2\omega) - \cosh(2k)}. \tag{25}$$

8. 3-Soliton and 4-Soliton Solutions

To obtain the 3-soliton solution in the bilinear form equation (11), first we put

$$\begin{aligned} F &= Z_1 + Z_2 + Z_3 + A_{123}Z_1Z_2Z_3, \\ G &= 1 + A_{12}Z_1Z_2 + A_{23}Z_2Z_3 + A_{31}Z_3Z_1, \end{aligned} \tag{26}$$

in which $Z_i = \exp(k_i n + \omega_i j + \delta_i)$, $(i = 1, 2, 3)$, $A_{i,j}$ is the coupling constant representing the interaction of two solitons and A_{123} is the constant for three solitons. Then the next four expressions (27)-(30) are required for the expression (26) to satisfy the equation (11);
the dispersion relations of each soliton,

$$\sinh \omega_i = \gamma \sinh k_i, \qquad (i = 1, 2, 3) \tag{27}$$

the coupling constants of two solitons,

$$A_{i,j} = -\frac{\cosh(\omega_i - \omega_j) - \cosh(k_i - k_j)}{\cosh(\omega_i + \omega_j) - \cosh(k_i + k_j)}, \tag{28}$$

the coupling constant of three solitons,

$$A_{123} = A_{12}A_{23}A_{31}, \tag{29}$$

and the additional identical relation,

$$A_{123}P_0 - A_{12}P_3 - A_{23}P_1 - A_{31}P_2 = 0, \tag{30}$$

where P_0, P_i are defined as

$$\begin{aligned} P_0 &= \sinh(\omega_1 + \omega_2 + \omega_3) - \gamma \sinh(k_1 + k_2 + k_3), \\ P_1 &= \sinh(\omega_2 + \omega_3 - \omega_1) - \gamma \sinh(k_2 + k_3 - k_1), \\ P_2 &= \sinh(\omega_3 + \omega_1 - \omega_2) - \gamma \sinh(k_3 + k_1 - k_2), \\ P_3 &= \sinh(\omega_1 + \omega_2 - \omega_3) - \gamma \sinh(k_1 + k_2 - k_3). \end{aligned} \tag{31}$$

The validity of the identical relation (30) under the condition of (27) is proved by using the symbolic-manipulations software Maple.[6)]

Next, we refer to the 4-soliton solution. We put in the equation (11)

$$\begin{aligned} F &= Z_1 + Z_2 + Z_3 + Z_4 + A_{123}Z_1Z_2Z_3 + A_{234}Z_2Z_3Z_4 + A_{341}Z_3Z_4Z_1 + A_{412}Z_4Z_1Z_2, \\ G &= 1 + A_{12}Z_1Z_2 + A_{13}Z_1Z_3 + A_{14}Z_1Z_4 + A_{23}Z_2Z_3 + A_{24}Z_2Z_4 + A_{34}Z_3Z_4 + A_{1234}Z_1Z_2Z_3Z_4, \end{aligned} \tag{32}$$

where $Z_i = \exp(k_i n + \omega_i j + \delta_i)$, $(i = 1, 2, 3, 4)$. Then the next four expressions (33)–(36) are required for the expression (32) to satisfy the equation (11);
the dispersion relations

$$\sinh \omega_i = \gamma \sinh k_i, \qquad (i = 1, 2, 3, 4), \tag{33}$$

the coupling constants of three solitons

$$A_{123} = A_{12}A_{13}A_{23}, \quad A_{234} = A_{23}A_{24}A_{34}, \quad A_{341} = A_{34}A_{31}A_{41}, \quad A_{412} = A_{41}A_{42}A_{12}, \tag{34}$$

the coupling constant of four solitons

$$A_{1234} = A_{12}A_{13}A_{14}A_{23}A_{24}A_{34}, \tag{35}$$

where $A_{i,j}$ is the coupling constant of two solitons defined by (28), and the additional identical relation

$$A_{1234}Q_0 + A_{123}Q_4 + A_{234}Q_1 + A_{341}Q_2 + A_{412}Q_3 + A_{12}A_{34}Q_{12} + A_{13}A_{24}Q_{13} + A_{14}A_{23}Q_{14} = 0, \tag{36}$$

where $Q_0,\ Q_i,\ Q_{i,j}$ are defined by

$$\begin{aligned} Q_0 &= \cosh(\Omega) - \cosh(K), \\ Q_i &= \cosh(\Omega - 2\omega_i) - \cosh(K - 2k_i), \\ Q_{i,j} &= \cosh(\Omega - 2\omega_i - 2\omega_j) - \cosh(K - 2k_i - 2k_j), \\ &\text{with} \quad \Omega = \sum_{i=1}^{4} \omega_i, \qquad K = \sum_{i=1}^{4} k_i. \end{aligned} \tag{37}$$

The validity of the identity (36) is also proved by using Maple.

9. N-Soliton Solution

In this section, we are developing the N-soliton solution of the equation. Although it is actually obtained by using the mathematical induction, we leave out the process and describe the result only.

We assume in the equation (11),

$$\begin{aligned} F &= \sum_{\ell=0}^{2\ell+1\le N} \left[\sum_{(i_1,i_2,\cdots,i_{2\ell+1})}^{(1,2,\cdots,N)} A_{i_1,i_2,\cdots,i_{2\ell+1}} Z_{i_1} Z_{i_2} \cdots Z_{i_{2\ell+1}} \right], \\ G &= 1 + \sum_{\ell=1}^{2\ell\le N} \left[\sum_{(i_1,i_2,\cdots,i_{2\ell})}^{(1,2,\cdots,N)} A_{i_1,i_2,\cdots,i_{2\ell}} Z_{i_1} Z_{i_2} \cdots Z_{i_{2\ell}} \right], \end{aligned} \tag{38}$$

in which the coupling constant of *one* soliton A_i is defined as $A_i \equiv 1$, the factor of each soliton Z_i is put $Z_i = \exp(k_i n + \omega_i j + \delta_i)$, $(i = 1,\ 2, \cdots,\ N)$, and the suffix $(i_1, i_2, \cdots, i_{2\ell+1})$ is a subset defined as an arbitrary $(2\ell + 1)$-combination from the set $(1, 2, \cdots, N)$ and the summation is carried out for all the subsets.

If we substitute the expression (38) into the equation (11), we get the dispersion relations of each soliton and the coupling constants of two solitons $A_{i,j}$ as the same forms (27) and (28) respectively.

The coupling constants for 3- or higher soliton are given as

$$A_{i_1,i_2,\cdots,i_\ell} = \Big[\prod_{k=1}^{\ell-1} A_{i_k,i_\ell}\Big] A_{i_1,i_2,\cdots,i_{\ell-1}}, \qquad (\ell = 3,\ 4,\ \cdots,\ N) \tag{39}$$

in the recurrence formula, namely $A_{i_1,i_2,\cdots,i_\ell}$ is the product of all the coupling constants of two solitons included in the suffix. The results obtained here are anticipated.

The additional identity in this case depends on the even-odd parity of the number N.

In the case of $N = 2n + 1, (n = 1, 2, \cdots)$, if we define the function with hyperbolic sine as

$$\begin{aligned} P_0 &= \sinh(\Omega) - \gamma \sinh(K), \\ P_{i_1,i_2,\cdots,i_\ell} &= \sinh(\Omega - 2\omega_{i_1} - 2\omega_{i_2} - \cdots - 2\omega_{i_\ell}) - \gamma \sinh(K - 2k_{i_1} - 2k_{i_2} - \cdots - k_{i_\ell}), \\ & \qquad (\ell = 1,\ 2,\ \cdots,\ n) \\ &\text{with} \qquad \Omega = \sum_{i=1}^{2n+1} \omega_i, \qquad K = \sum_{i=1}^{2n+1} k_i, \end{aligned} \tag{40}$$

the additional identity is obtained as

$$A_{1,2,\cdots,2n+1} P_0 + \sum_{\ell=1}^{n} (-)^\ell \sum_{(i_1,i_2,\cdots,i_\ell)}^{(1,2,\cdots,2n+1)} A_{\overline{i_1,i_2,\cdots,i_\ell}} A_{i_1,i_2,\cdots,i_\ell} P_{i_1,i_2,\cdots,i_\ell} = 0, \tag{41}$$

where the over-line in the suffix means the complement of the numbers of the suffix for the set $(1, 2, \cdots, 2n+1)$, *e.g.* $A_{\overline{1}} = A_{2,3,\cdots,2n+1}$, $A_{\overline{1,2}} = A_{3,4,\cdots,2n+1}$, *etc.*

In the case of $N = 2n+2$, $(n = 1, 2, \cdots)$, if we define the function with hyperbolic cosine as

$$\begin{aligned} Q_0 &= \cosh(\Omega) - \cosh(K), \\ Q_{i_1,i_2,\cdots,i_\ell} &= \cosh(\Omega - 2\omega_{i_1} - 2\omega_{i_2} - \cdots - 2\omega_{i_\ell}) - \cosh(K - 2k_{i_1} - 2k_{i_2} - \cdots - 2k_{i_\ell}), \\ & \qquad\qquad (\ell = 1,\ 2,\ \cdots,\ n+1) \end{aligned} \tag{42}$$

$$\text{with} \qquad \Omega = \sum_{i=1}^{2n+2} \omega_i, \qquad K = \sum_{i=1}^{2n+2} k_i,$$

the additional identity is given as

$$A_{1,2,\cdots,2n+2} Q_0 + \sum_{\ell=1}^{n} \sum_{(i_1,i_2,\cdots,i_\ell)}^{(1,2,\cdots,2n+2)} A_{\overline{i_1,i_2,\cdots,i_\ell}} A_{i_1,i_2,\cdots,i_\ell} Q_{i_1,i_2,\cdots,i_\ell} + \frac{1}{2} \sum_{(i_1,i_2,\cdots,i_{n+1})}^{(1,2,\cdots,2n+2)} A_{\overline{i_1,i_2,\cdots,i_{n+1}}} A_{i_1,i_2,\cdots,i_{n+1}} Q_{i_1,i_2,\cdots,i_{n+1}} = 0, \tag{43}$$

where the over-line means the complement of the numbers of the suffix for the set $(1, 2, \cdots, 2n+2)$.

The identity (41) for $n = 1$ coincides with the one (30) for 3-soliton, and also the identity (43) for $n = 1$ coincides with the one (36) for 4-soliton.

We try to prove the validity of the identity in the case of $N = 5$ by using Maple, but we can not help giving it up for the memory shortage of the computer. So the validity of these identities for $N \geq 5$ are not proved yet.

10. Conclusions

In this paper, we proposed a new 1st order difference-difference nonlinear equation, and presented some exact solutions, namely from the 1-soliton solution up to the 4-soliton one. And the equation is considered to be the simplest of those that have the mKdV type soliton solution.

We obtained the definite expression of the N-soliton solution with its additional identity, but the direct proof of the validity of the identity for $N \geq 5$ is very difficult as mentioned above. However, we are convinced that our assumption is correct, and there seems to be a possibility that our proposed equation will belong to an integrable system.

It is also noticed that the solutions obtained here have close similarities to the ones of the discrete time Toda lattice equation. Therefore, further investigations are needed for clearer relations between these equations.

Acknowledgement

We would like to express our gratitude to Dr. Chikage Habe, who first proved the validity of the identity (30) by using a symbolic-manipulations software.

References

1) R. Hirota: *Discrete Analogue of a Generalized Toda Equation,* J. Phys. Soc. Jpn. **50**, (1981), 3785.
2) R. Hirota: *Direct Method in Soliton Theory,* (Iwanami Shoten, Tokyo, 1992) [in Japanese].
3) R. Hirota: *Nonlinear Partial Difference Equation I-V,* J.Phys.Soc.Jpn. **43**, 1423, 2074, 2079; **45**, 321; **46**, 312 (1977-8).
4) K. Maruno, K. Kajiwara, S. Nakao and M. Oikawa, Phys. Lett. A, **229**, (1997), 173.
5) S. Saito and N. Saitoh: *A Characterization of Discrete Time Soliton Equations*, J. Phys. Soc. Jpn. **70**, (2001), 3517.
6) *Maple* is the registered trademark of Waterloo Maple Inc., Canada