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Algebraic Properties of Singular Integral Operators on L^2 with Cauchy Kernel

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This paper is dedicated to the memory of late Professor Takayuki Furuta

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Keywords. Singular integral operator, Toeplitz operator, Hardy space, hyponormal operator.

Abstract. Let α and β be functions in $L^\infty(\mathbb{T})$, where \mathbb{T} is the unit circle. Let P denote the orthogonal projection from $L^2(\mathbb{T})$ onto the Hardy space $H^2(\mathbb{T})$, and $Q = I - P$, where I is the identity operator on $L^2(\mathbb{T})$. This paper is concerned with the singular integral operators $S_{\alpha,\beta}$ on $L^2(\mathbb{T})$ of the form $S_{\alpha,\beta}f = \alpha Pf + \beta Qf$, for $f \in L^2(\mathbb{T})$. In this paper, we study the hyponormality of $S_{\alpha,\beta}$ which is related to the Toeplitz operator on $H^2(\mathbb{T})$.

1. Introduction

For $1 \leq p \leq \infty$, $L^p = L^p(\mathbb{T})$ denotes the usual Lebesgue space on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and $H^p = H^p(\mathbb{T})$ denotes the usual Hardy space on \mathbb{T} . If $p = 2$, then $\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{ix}) \bar{g}(e^{ix}) dx$ and $\|f\| = \|f\|_2$. Let $z = e^{ix}$, let $H_0^2 = zH^2$, and let $H^{2\perp} = L^2 \ominus H^2$. Then $H^{2\perp} = \overline{H_0^2}$. Let P denote the orthogonal projection of L^2 onto H^2 . Let I denote the identity operator on L^2 , and let $Q = I - P$. Then Q is an orthogonal projection of L^2 onto $H^{2\perp}$. In L^2 , the sequence e_n , defined as $e_n(e^{ix}) = e^{inx}$, $n \in \mathbb{Z}$, is an orthonormal sequence. Here the n -th Fourier coefficient of f is defined by $\langle f, e_n \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{ix}) e^{-inx} dx = \hat{f}(n) = f_n$. Let P_0 denote the rank one orthogonal projection of L^2 onto \mathbb{C} such that $(P_0 f)(z) = \hat{f}(0)(f \in L^2)$. Let $I_0 = I - P_0$. For $\alpha \in L^\infty$, let M_α denote the

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multiplication operator on L^2 such that $M_\alpha f = \alpha f$, ($f \in L^2$), let T_α denote the Toeplitz operator on H^2 such that

$$T_\alpha f = P(\alpha f), (f \in H^2),$$

let \tilde{T}_α denote the operator on $H^{2\perp}$ such that

$$\tilde{T}_\alpha f = Q(\alpha f), (f \in H^{2\perp}),$$

let H_α denote the Hankel operator of H^2 to $H^{2\perp}$ such that

$$H_\alpha f = Q(\alpha f), (f \in H^2)$$

and let \tilde{H}_α denote the operator on $H^{2\perp}$ to H^2 such that

$$\tilde{H}_\alpha f = P(\alpha f), (f \in H^{2\perp}).$$

Then $\tilde{H}_\alpha = H_\alpha^*$. For $\alpha, \beta \in L^\infty$, let $S_{\alpha, \beta}$ denote the singular integral operator on L^2 such that

$$S_{\alpha, \beta} f = \alpha P f + \beta Q f, (f \in L^2).$$

Then

$$(S_{\alpha, \beta} f)(z) = \frac{\alpha(z) + \beta(z)}{2} f(z) + \frac{\alpha(z) - \beta(z)}{2} \frac{1}{\pi i} \int_{\mathbb{T}} \frac{f(\tau)}{\tau - z} d\tau,$$

where the integral is understood in the sense of Cauchy's principal value (cf. [6], p.12). If $f \in L^1$, then $(S_{\alpha, \beta} f)(z)$ exists for almost all $z \in \mathbb{T}$. The normality of $S_{\alpha, \beta}$ was established by Nakazi and the author [27]. An operator A is called hyponormal if its self-commutator $[A^*, A] = A^* A - A A^*$ is positive. When $\alpha - \beta$ is a constant, then $S_{\alpha, \beta}$ is hyponormal if and only if $S_{\alpha, \beta}$ is normal ([13]). In this paper, we study the hyponormal operator $S_{\alpha, \beta}$.

2. HYPONORMAL SI-OPERATOR

In this section, when β is a complex number, the conditions of symbols α and β of hyponormal operators $S_{\alpha, \beta}$ are determined using Toeplitz operators and Hankel operators.

Lemma 1.1. *Let α and β be in L^∞ . Suppose $S_{\alpha, \beta}$ is a hyponormal operator.*

- (1) *If $\bar{\alpha}$ is in H^∞ , then $\bar{\beta}$ is in H^∞ , and for all $f_2 \in H^{2\perp}$, $\|\tilde{H}_{\bar{\alpha}} f_2\| \leq \|\tilde{H}_{\bar{\beta}} f_2\|$.*
- (2) *If β is in H^∞ , then α is in H^∞ , and for all $f_1 \in H^2$, $\|H_{\bar{\beta}} f_1\| \leq \|H_{\bar{\alpha}} f_1\|$.*

Proof. For all f in L^2 , $S_{\alpha, \beta}^* f = P(\bar{\alpha} f) + Q(\bar{\beta} f)$. Since $S_{\alpha, \beta}$ is hyponormal, it follows that for all $f_1 \in H^2$ and $f_2 \in H^{2\perp}$,

$$\begin{aligned}
 0 &\leq \langle (S_{\alpha,\beta}^* S_{\alpha,\beta} - S_{\alpha,\beta} S_{\alpha,\beta}^*)(f_1 + f_2), f_1 + f_2 \rangle \\
 &= \|S_{\alpha,\beta}(f_1 + f_2)\|^2 - \|S_{\alpha,\beta}^*(f_1 + f_2)\|^2 \\
 &= \|\alpha f_1 + \beta f_2\|^2 - \|P\bar{\alpha}(f_1 + f_2)\|^2 - \|Q\bar{\beta}(f_1 + f_2)\|^2.
 \end{aligned}$$

Therefore, for all $f_1 \in H^2$,

$$\begin{aligned}
 0 &\leq \|\bar{\alpha}f_1\|^2 - \|P\bar{\alpha}f_1\|^2 - \|Q\bar{\beta}f_1\|^2 \\
 &= \|Q\bar{\alpha}f_1\|^2 - \|Q\bar{\beta}f_1\|^2,
 \end{aligned}$$

and for all $f_2 \in H^{2\perp}$,

$$\begin{aligned}
 0 &\leq \|\bar{\beta}f_2\|^2 - \|P\bar{\alpha}f_2\|^2 - \|Q\bar{\beta}f_2\|^2 \\
 &= \|P\bar{\beta}f_2\|^2 - \|P\bar{\alpha}f_2\|^2.
 \end{aligned}$$

Suppose $\bar{\alpha}$ is in H^∞ . Since for all $f_1 \in H^2$, $\|Q\bar{\beta}f_1\| \leq \|Q\bar{\alpha}f_1\|$, this implies that $Q\bar{\beta}f_1 = 0$, and hence $\bar{\beta}$ is in H^∞ . Hence (1) holds.

Suppose β is in H^∞ . Since for all $f_2 \in H^{2\perp}$, $\|P\bar{\alpha}f_2\| \leq \|P\bar{\beta}f_2\|$, this implies that $P\bar{\alpha}f_2 = 0$, and hence α is in H^∞ . Hence (2) holds. \square

Lemma 1.2. *Let α be in L^∞ , and let β be a complex number. Then for all $f_1 \in H^2$ and $f_2 \in H^{2\perp}$,*

$$(S_{\alpha,\beta}^* S_{\alpha,\beta} - S_{\alpha,\beta} S_{\alpha,\beta}^*)(f_1 + f_2) = P|\alpha|^2 f_1 - \alpha P\bar{\alpha} f_1 + (\beta - \alpha)P\bar{\alpha} f_2 + \bar{\beta}Q\alpha f_1.$$

Proof. Let $A = S_{\alpha,\beta}$. Then

$$\begin{aligned}
 A^* A(f_1 + f_2) &= A^*(\alpha f_1 + \beta f_2) = A^*(\alpha f_1) + A^*(\beta f_2) \\
 &= P\bar{\alpha}\alpha f_1 + Q\bar{\beta}\alpha f_1 + P\bar{\alpha}\beta f_2 + Q\bar{\beta}\beta f_2 \\
 &= P|\alpha|^2 f_1 + \bar{\beta}Q\alpha f_1 + \beta P\bar{\alpha} f_2 + |\beta|^2 f_2,
 \end{aligned}$$

and

$$\begin{aligned}
 AA^*(f_1 + f_2) &= AP\bar{\alpha}(f_1 + f_2) + AQ\bar{\beta}(f_1 + f_2) \\
 &= \alpha P\bar{\alpha}(f_1 + f_2) + \beta Q\bar{\beta}(f_1 + f_2) \\
 &= \alpha P\bar{\alpha} f_1 + \alpha P\bar{\alpha} f_2 + |\beta|^2 f_2.
 \end{aligned}$$

Hence

$$(A^* A - AA^*)(f_1 + f_2) = P|\alpha|^2 f_1 - \alpha P\bar{\alpha} f_1 + (\beta - \alpha)P\bar{\alpha} f_2 + \bar{\beta}Q\alpha f_1.$$

\square

Theorem 1.1. *Let α be in L^∞ and let β be a complex number. Then $S_{\alpha,\beta}$ is hyponormal if and only if T_α is analytic.*

Proof. Suppose $S_{\alpha,\beta}$ is hyponormal. Since β is a complex number, it follows from Lemma 1.1 (2), that α is in H^∞ , and hence T_α is analytic. Conversely suppose T_α is analytic. Then α is in H^∞ . Let $A = S_{\alpha,\beta}$. By Lemma 1.2, for all $f_1 \in H^2$ and $f_2 \in H^{2\perp}$,

$$(A^*A - AA^*)(f_1 + f_2) = P|\alpha|^2 f_1 - \alpha P\bar{\alpha} f_1.$$

Hence

$$\begin{aligned} \langle (A^*A - AA^*)(f_1 + f_2), f_1 + f_2 \rangle &= \langle P|\alpha|^2 f_1, f_1 + f_2 \rangle - \langle \alpha P\bar{\alpha} f_1, f_1 + f_2 \rangle \\ &= \langle P|\alpha|^2 f_1, f_1 \rangle - \langle \alpha P\bar{\alpha} f_1, f_1 \rangle \\ &= \|\bar{\alpha} f_1\|^2 - \|P\bar{\alpha} f_1\|^2 = \|Q\bar{\alpha} f_1\|^2 \geq 0. \end{aligned}$$

Therefore $S_{\alpha,\beta}$ is hyponormal. □

Corollary 1.1. *Let φ be in L^∞ . Then $S_{\varphi,0} = M_\varphi P$ is hyponormal if and only if $S_{\varphi,1} = \varphi P + Q$ is hyponormal if and only if T_φ is analytic.*

Suppose α is a constant multiple of a unimodular function in L^∞ and β is a complex number. Then we study the conditions of symbols α and β of subnormal and quasinormal $S_{\alpha,\beta}$.

Lemma 1.3. ([13]) *For a bounded analytic function φ , the Toeplitz operator T_φ is quasinormal if and only if φ is a constant multiple of an inner function.*

Theorem 1.2. *Let α be a constant multiple of a unimodular function in L^∞ and let β be a complex number. Then $S_{\alpha,\beta}$ is subnormal if and only if $S_{\alpha,\beta}$ is hyponormal if and only if $S_{\alpha,\beta}$ is quasinormal if and only if T_α is analytic and quasinormal if and only if α is a constant multiple of an inner function.*

Proof. Let $A = S_{\alpha,\beta}$. Suppose A is subnormal. Since every subnormal operator is hyponormal, it follows that A is hyponormal. By Lemma 1.1(2), this implies that α is in H^∞ . Since $|\alpha|$ is a constant, it follows that α is a constant multiple of an inner function. By Lemma 1.3, T_α is quasinormal. Conversely suppose T_α is analytic and quasinormal. By Lemma 1.3, this implies that α is a constant multiple of an inner function. By the proof of Lemma 1.2, for all $f_1 \in H^2$ and $f_2 \in H^2$,

$$\begin{aligned} A^*A(f_1+f_2) &= P|\alpha|^2f_1 + \bar{\beta}Q\alpha f_1 + \beta P\bar{\alpha}f_2 + |\beta|^2f_2 \\ &= |\alpha|^2f_1 + |\beta|^2f_2. \end{aligned}$$

Since α is a constant multiple of an inner function, it follows that

$$\begin{aligned} (A(A^*A) - (A^*A)A)(f_1+f_2) &= A(A^*A)(f_1+f_2) - (A^*A)(\alpha f_1 + \beta f_2) \\ &= A(|\alpha|^2f_1 + |\beta|^2f_2) - (|\alpha|^2\alpha f_1 + |\beta|^2\beta f_2) \\ &= |\alpha|^2\alpha f_1 + |\beta|^2\beta f_2 - (|\alpha|^2\alpha f_1 + |\beta|^2\beta f_2) = 0. \end{aligned}$$

Hence A is quasinormal. We recall that every quasinormal operator is subnormal. Hence A is subnormal. By Theorem 1.1, this completes the proof. \square

Suppose φ is a constant multiple of a unimodular function in L^∞ . Then we study the conditions of symbols φ of 2-contractive (i.e. convex, c.f. [1], [3]) operators $S_{\varphi,0} = M_\varphi P$.

Lemma 1.4. *Let α and β be in L^∞ . Suppose $S_{\alpha,\beta}$ is 2-contractive (i.e. convex).*

- (1) *If $|\alpha| \geq 1$ a.e., then for all f_1 in H^2 , $\|(\alpha T_\alpha + \beta H_\alpha)f_1\| \geq \|f_1\|$.*
- (2) *If $|\beta| \geq 1$ a.e., then for all f_2 in $H^{2\perp}$, $\|(\alpha \tilde{H}_\beta + \beta \tilde{T}_\beta)f_2\| \geq \|f_2\|$.*
- (3) *If α is a bounded analytic function, then for all f_1 in H^2 , $\|f_1\|^2 - 2\|\alpha f_1\|^2 + \|\alpha^2 f_1\|^2 \geq 0$.*
- (4) *If $\bar{\beta}$ is a bounded analytic function, then for all f_2 in $H^{2\perp}$, $\|f_2\|^2 - 2\|\beta f_2\|^2 + \|\beta^2 f_2\|^2 \geq 0$.*

Proof. (1): Let $A = S_{\alpha,\beta}$. Then A is 2-contractive (i.e. convex). For all f_1 in H^2 and f_2 in $H^{2\perp}$,

$$\|f_1 + f_2\|^2 - 2\|A(f_1 + f_2)\|^2 + \|A^2(f_1 + f_2)\|^2 \geq 0.$$

Hence

$$\|f_1\|^2 - 2\|Af_1\|^2 + \|A^2f_1\|^2 \geq 0.$$

Since $A(f_1 + f_2) = \alpha f_1 + \beta f_2$ and

$$A^2(f_1 + f_2) = A(\alpha f_1 + \beta f_2) = \alpha P\alpha f_1 + \alpha P\beta f_2 + \beta Q\alpha f_1 + \beta Q\beta f_2,$$

it follows that

$$\begin{aligned}
 0 &\leq \|f_1\|^2 - 2\|Af_1\|^2 + \|A^2f_1\|^2 \\
 &= \|f_1\|^2 - 2\|\alpha f_1\|^2 + \|\alpha P\alpha f_1 + \beta Q\alpha f_1\|^2 \\
 &\leq \|\alpha P\alpha f_1 + \beta Q\alpha f_1\|^2 - \|f_1\|^2 \\
 &= \|(\alpha T_\alpha + \beta H_\alpha)f_1\|^2 - \|f_1\|^2.
 \end{aligned}$$

(2): Since A is 2-contractive (i.e. convex), it follows that for all f_2 in $H^{2\perp}$,

$$\|f_2\|^2 - 2\|Af_2\|^2 + \|A^2f_2\|^2 \geq 0.$$

Hence

$$\begin{aligned}
 0 &\leq \|f_2\|^2 - 2\|Af_2\|^2 + \|A^2f_2\|^2 \\
 &= \|f_2\|^2 - 2\|\beta f_2\|^2 + \|\alpha P\beta f_2 + \beta Q\beta f_2\|^2 \\
 &\leq \|\alpha P\beta f_2 + \beta Q\beta f_2\|^2 - \|f_2\|^2 \\
 &= \|(\alpha \tilde{H}_\beta + \beta \tilde{T}_\beta)f_2\|^2 - \|f_2\|^2.
 \end{aligned}$$

(3): Since A is 2-contractive (i.e. convex), it follows that for all f_1 in H^2 ,

$$\begin{aligned}
 0 &\leq \|f_1\|^2 - 2\|Af_1\|^2 + \|A^2f_1\|^2 \\
 &= \|f_1\|^2 - 2\|\alpha f_1\|^2 + \|\alpha^2 f_1\|^2.
 \end{aligned}$$

(4): Since A is 2-contractive (i.e. convex), it follows that for all f_2 in $H^{2\perp}$,

$$\begin{aligned}
 0 &\leq \|f_2\|^2 - 2\|Af_2\|^2 + \|A^2f_2\|^2 \\
 &= \|f_2\|^2 - 2\|\beta f_2\|^2 + \|\beta^2 f_2\|^2.
 \end{aligned}$$

□

Theorem 1.3. Let φ be a constant multiple of a unimodular function in L^∞ . Suppose an operator $S_{\varphi,0} = M_\varphi P$ is 2-contractive (i.e. convex, c.f. [1], [3]). Then $|\varphi| \geq 1$ a.e. and $|\varphi| \cdot \|T_\varphi f_1\| \geq \|f_1\|$ for all f_1 in H^2 .

Proof. Let $A = S_{\varphi,0}$. Since A is 2-contractive (i.e. convex), it follows from Lemma 1.4(1), for all f_1 in H^2 , $\frac{1}{|\varphi|} \|f_1\| \leq \|T_\varphi f_1\| \leq |\varphi| \cdot \|f_1\|$. Hence $|\varphi| \geq 1$ a.e. □

Definition 1.1. For $0 < p < \infty$, A belongs to class $B(p)$ if $(A^*A)^p = A^{*p}A^p$.

By the elementary calculation in the proof of the following corollary, it follows that if A is contractive and belongs to class $B(2)$, then A is 2-contractive.

Corollary 1.2. *Let φ be a unimodular function in L^∞ . Suppose $S_{\varphi,0} = M_\varphi P$ is quasinormal. Then $S_{\varphi,0}$ is 2-contractive (i.e. convex), $S_{\varphi,0}$ is contractive and belongs to class $B(2)$.*

Proof. Suppose $A = S_{\varphi,0}$ is quasinormal. By Theorem 1.2, φ is an inner function. For all f in L^2 , $\|Af\| = \|\varphi Pf\| = \|Pf\| \leq \|f\|$. Therefore A is contractive. Since $A(A^*A) = (A^*A)A$, it follows that $(A^*A)^2 = A^*A^2$, and hence A is contractive and belongs to class $B(2)$. Suppose A is contractive and belongs to class $B(2)$. Then $I - A^*A$ is a positive operator. Hence, for all f in L^2 ,

$$\begin{aligned} \langle (I - 2A^*A + A^*A^2)f, f \rangle &= \langle (I - 2A^*A + (A^*A)^2)f, f \rangle \\ &= \langle (I - A^*A)^2f, f \rangle \\ &= \langle (I - A^*A)f, (I - A^*A)f \rangle \\ &= \|(I - A^*A)f\|^2 \geq 0. \end{aligned}$$

Therefore A is 2-contractive (i.e. convex). □

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