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On Some Singular Integral Operators Which are One to One Mappings on the Weighted Lebesgue-Hilbert Spaces

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Dedicated to Professor Takahiko Nakazi on the occasion of his 70th birthday

ABSTRACT

Let ϕ be a bounded measurable function on the unit circle. Then we shall give the form of a weight W for which the singular integral operator $\phi P_+ + P_-$ is left invertible in the weighted space $L^2(W)$. P_+ is an analytic projection, P_- is a co-analytic projection. When W is an (A_2) weight, $\phi P_+ + P_-$ is left invertible (resp. invertible) in $L^2(W)$ if and only if Toeplitz operator T_ϕ is left invertible (resp. invertible) in $H^2(W)$.

KEYWORDS: Singular integral operator, Riesz projection, Hardy space

MSC (2010): Primary 46J15, 47B35.

§ 1. INTRODUCTION.

Let m denote the normalized Lebesgue measure on the unit circle $\mathbf{T} = \{\zeta; |\zeta|=1\}$ and let χ denote the identity function on \mathbf{T} . For a function f in $L^1(m)$, its k -th Fourier coefficient $\hat{f}(k)$ is defined by

$$\hat{f}(k) = \int_{\mathbf{T}} \chi^{-k} f dm$$

for all integers k . For a function f in $L^1(m)$, its harmonic conjugate function \tilde{f} is defined by the singular integral

$$\tilde{f}(\theta) = VP \int_{\mathbf{T}} f(\theta - t) \cot \frac{t}{2} dm(t).$$

Let $C(\mathbf{T})$ be an algebra of all continuous functions f on \mathbf{T} , and let A be a disc algebra of all functions f in $C(\mathbf{T})$ such that $\hat{f}(k) = 0$ for all negative integers k . The Hardy spaces H^p , $0 < p \leq \infty$, are defined as follows. For $0 < p < \infty$, H^p is the $L^p(m)$ -closure of A , while H^∞ is defined to be the

weak-* closure of A in $L^\infty(m)$. If an f in H^p has the form $f = \exp(u + i\bar{u} + ic)$ a.e. for some function u in $L^k(m)$ and some real constant c , then f is called an outer function. Let A_0 be the subspace of all functions f in A which satisfy $\hat{f}(0) = 0$, and let \bar{A}_0 be the subspace of all complex conjugate functions of functions in A_0 . Since the intersection of H^1 and \bar{H}_0^1 is only the zero function, the analytic projection P_+ is defined as

$$P_+(f_1 + f_2) = f_1, \text{ for all } f_1 \text{ in } H^1 \text{ and all } f_2 \text{ in } \bar{H}_0^1.$$

The co-analytic projection P_- is defined by $P_- = I - P_+$ where I is an identity operator on $H^1 + \bar{H}_0^1$. Then

$$P_\pm f = \frac{1}{2}\{f \pm i\tilde{f} \pm \hat{f}(0)\}, \text{ for all } f \text{ in } A + \bar{A}_0.$$

For a ϕ in $L^\infty(m)$, the Toeplitz operator T_ϕ is defined as a map from H^2 to H^2 by

$$T_\phi f = P_+(\phi f), \text{ for all } f \text{ in } H^2.$$

A non-negative integrable function W on \mathbf{T} is said to be a weight. P_+ is bounded on $L^p(W)$ if and only if W satisfies the A_p -condition (cf.[6], p.254). (A_p) denotes the set of all positive weights W satisfying the A_p -condition. In the case $p=2$, Helson-Szegö theorem gives the form of a weight W in (A_2) (cf.[6], p.147 and [7]). If W is in (A_2) , then T_ϕ is bounded in $H^2(W)$ and $\phi P_+ + P_-$ is bounded in $L^2(W)$. A weight W does not necessarily belong to (A_2) when those operators are bounded. In this paper we shall give the form of a weight W such that $\phi P_+ + P_-$ is bounded and left invertible in $L^2(W)$. It should be mentioned that W is in (A_2) if and only if there exist a function k in H^1 and a constant $\rho, \rho < 1$ such that $|W - k| \leq \rho W$ a.e.. If W is in (A_2) , then $\log W$ is in $BMO = L_R^\infty(m) + \tilde{L}_R^\infty(m)$.

Definition. (1) For a function λ in $L^\infty(m)$,

$$\begin{aligned} A(\lambda) &= \{s \in BMO; \lambda = |\lambda| \exp(is) \text{ a.e.}\}, \\ A &= \{\lambda \in L^\infty(m); |\lambda| \exp(\tilde{s}) \text{ is bounded for some } s \text{ in } A(\lambda)\}. \end{aligned}$$

(2) For a function ϕ in $L^\infty(m)$, we shall write

$$E(\phi) = \{\zeta \in \mathbf{T}; \phi(\zeta) \neq 1\} \text{ and } m(E(\phi)) = \int_E dm = \int_E \frac{dt}{2\pi}.$$

$I(\phi, +), I(\phi, -)$ denote intervals such that

$$\begin{aligned} I(\phi, +) &= [\max\{1, \|\phi\|_\infty\}, \infty), \\ I(\phi, -) &= (0, \min\{1, \text{ess inf } |\phi|\}] \text{ and put} \end{aligned}$$

$$I(\phi) = I(\phi, +) \cup I(\phi, -),$$

$$J(\phi) = \{t \in I(\phi); t^2 - \phi \text{ belongs to } \Lambda\}.$$

(3) For a function ϕ in $L^\infty(m)$ and a constant t in $I(\phi)$ satisfying $m\{\phi = t^2\} = 0$, put

$$r(t, \phi) = \left| \frac{(\phi - 1)t}{t^2 - \phi} \right|,$$

and for a function v satisfying $|v| \leq \cos^{-1} r(t, \phi)$ a.e., put

$$U(t, \phi, v) = \cosh^{-1} \left(\frac{\cos v}{r(t, \phi)} \right).$$

In this paper we shall assume $-\pi \leq \text{Arg } z < \pi$. For any ϕ in $L^\infty(m)$, $0 \leq m(E(\phi)) \leq 1$. If $|\phi| = 1$ a.e., then $I(\phi) = (0, \infty)$. For any λ in $L^\infty(m)$, $\text{Arg } \lambda$ belongs to a set $A(\lambda)$. $\Lambda \cdot \Lambda = \Lambda$ and Λ contains a set $\exp H^\infty$. λ belongs to Λ if and only if there exist two functions t, s in $L^\infty(m)$ such that $t + \bar{s}$ is bounded above and $\lambda = \exp(t + is)$ a.e.. The following Lemma A is useful to study the boundedness and the left invertibility of $\phi P_+ + P_-$ in $L^2(W)$.

Lemma A. *Suppose ϕ is a function in $L^\infty(m)$ such that $m(E(\phi)) > 0$. Suppose t is a constant in $J(\phi)$ such that $m\{\phi = t^2\} = 0$. Then $r(t, \phi) \leq 1$ a.e.. For a weight W such that $\log W$ is integrable, the following conditions are equivalent.*

(i) *There exists a function k in H^1 such that*

$$|(t^2 - \phi)W - k| \leq \{1 - r(t, \phi)^2\}^{1/2} |t^2 - \phi|W \text{ a.e..}$$

(ii) *There exist three functions u, v, s , and a constant c such that*

$$|v| \leq \cos^{-1} r(t, \phi) \text{ a.e., and } m\{|v| = \pi/2\} = 0;$$

$$|u| \leq U(t, \phi, v) \text{ a.e. on } E(\phi), \text{ and } -\log(2 \cos v) \leq u \text{ a.e. on } E(\phi)^c;$$

$$s \text{ is in } A(t^2 - \phi), \text{ and } W = \left(\chi_{E(\phi)^c} \frac{t}{|t^2 - 1|} + \chi_{E(\phi)} \frac{1}{|\phi - 1|} \right) \exp(u - \bar{v} - \bar{s} - c) \text{ a.e..}$$

If $m(E(\phi)^c) > 0$ then $t \neq 1$. If W satisfies one of these conditions, then W^{-1} is integrable.

For a given function ϕ in $L^\infty(m)$, the form of a weight W such that $\phi P_+ + P_-$ is bounded in $L^2(W)$ was given in our preceding paper [14]. The proof of Lemma A is similar to it. In § 2, we shall give the proof. It is known that T_ϕ is left invertible (resp. invertible) in H^2 if and only if $\phi P_+ + P_-$ is left invertible (resp. invertible) in $L^2(m)$ (cf [10], p.71 and [15], p.393). Left invertibilities of singular integral operators $\phi P_+ + P_-$ and Toeplitz operators T_ϕ in weighted spaces were never been studied. In § 3, we shall give the form of a weight W such that $\phi P_+ + P_-$ (resp. T_ϕ) is bounded

and left invertible in $L^2(W)$ (resp. $H^2(W)$). A central role is played by the Cotlar-Sadosky lifting theorem and Lemma A. The invertibility of T_ϕ in weighted spaces was already studied by Rochberg [16]. In § 4, we shall consider the invertibility of $\phi P_+ + P_-$ and T_ϕ in weighted spaces. For a function f in $L^2(W)$, the $L^2(W)$ norm of f is denoted by $\|f\|_W = \left\{ \int_{\mathbb{T}} |f|^2 W dm \right\}^{1/2}$.

§ 2. PROOF OF LEMMA A.

We shall show that (i) implies (ii). Suppose $k=0$ in (i), then by the calculation we have $\phi=1$ a.e. which contradicts to $m(E(\phi))>0$. Hence we have $k \neq 0$. Since t is in $I(\phi)$, we have

$$|t^2 - \phi|^2 - |\phi - 1|^2 t^2 = (t^2 - 1)(t^2 - |\phi|^2) \geq 0 \text{ a.e.}$$

Hence $r(t, \phi) \leq 1$ a.e. and $|k| \leq 2W|t^2 - \phi|$ a.e.. Suppose $m(E(\phi)^c) > 0$ and $t=1$ in (i), then $k=0$ a.e. on $E(\phi)^c$ and hence $k=0$ a.e.. This contradiction implies that if $m(E(\phi)^c) > 0$ then $t \neq 1$. Since t is in $J(\phi)$, $t^2 - \phi$ belongs to Λ . Hence there exists a function s in $L^\infty(m)$ such that $t^2 - \phi = |t^2 - \phi| \exp(is)$ a.e. and $|t^2 - \phi| \exp(\bar{s})$ is bounded. Put $g = k \exp(\bar{s} - is)$, then $|g| \leq 2W|t^2 - \phi| \exp(\bar{s})$ a.e.. Hence g is a non-zero function in H^1 . Put $v = \text{Arg } g$, then $|v| \leq \cos^{-1} r(t, \phi)$ a.e. since

$$v = \text{Arg} \left(\frac{k}{t^2 - \phi} \right) \text{ a.e., and } \left| W - \frac{k}{t^2 - \phi} \right| \leq \{1 - r(t, \phi)^2\}^{1/2} W \text{ a.e.}$$

Since g is an outer function such that $\text{Re } g \geq 0$ a.e. and

$$\frac{\exp(iv - \bar{v})}{|\exp(iv - \bar{v})|} = \frac{g}{|g|} \text{ a.e.,}$$

there exists a positive constant γ such that $\exp(iv - \bar{v}) = \gamma g$ a.e. (cf.[11], Proposition 5). Put

$$u = \bar{v} + \bar{s} + \log W + \log t + \log \gamma + \chi_{E(\phi)} \log |\phi - 1| + \chi_{E(\phi)^c} \log |t - t^{-1}|,$$

then

$$W = \left(\chi_{E(\phi)^c} \frac{t}{|t^2 - 1|} + \chi_{E(\phi)} \frac{1}{|\phi - 1|} \right) \exp(u - \bar{v} - \bar{s} - c) \text{ a.e.}$$

Since $|1 - r(t, \phi) \exp(iv - u)|^2 \leq 1 - r(t, \phi)^2$ a.e. on $E(\phi)$, we have

$$e^{2u} - 2 \left(\frac{\cos v}{r(t, \phi)} \right) e^u + 1 \leq 0 \text{ a.e. on } E(\phi),$$

and hence $|u| \leq U(t, \phi, v)$ a.e. on $E(\phi)$. Since

$$\left| \frac{k}{t^2 - \phi} \right|^2 \leq 2W \text{Re} \left(\frac{k}{t^2 - \phi} \right) \text{ a.e.,}$$

we have

$$\left| \frac{k}{t^2 - \phi} \right| \leq 2W \cos v \text{ a.e.}$$

Hence $W^{-1} \leq 2\gamma |t^2 - \phi| \exp(\bar{v} + \bar{s}) \cos v$ a.e. Since $|v| \leq \pi/2$ a.e., $\exp(\bar{v}) \cos v$ is integrable (cf.[6], p. 161). Since t is in $J(\phi)$, W^{-1} is integrable. Since $(\cos v)^{-p}$ is integrable for some $p, p > 0$, we have $m\{|v| = \pi/2\} = 0$. Since

$$2\gamma W |t^2 - 1| \exp(\bar{v} + \bar{s}) \cos v \geq 1 \text{ a.e. on } E(\phi)^c,$$

we have $-\log(2 \cos v) \leq u$ a.e. on $E(\phi)^c$. We shall show that (ii) implies (i). Since $|u| \leq U(t, \phi, v)$ a.e. on $E(\phi)$, we have

$$|1 - r(t, \phi) \exp(iv - u)|^2 - (1 - r(t, \phi)^2) = r(t, \phi)^2 \left\{ e^{-2u} - 2 \left(\frac{\cos v}{r(t, \phi)} \right) e^{-u} + 1 \right\} \leq 0 \text{ a.e. on } E(\phi).$$

Put $k = t \exp\{i(v + s) - (v + s) - c\}$, then

$$|(t^2 - \phi)W - k| = |1 - r(t, \phi) \exp(iv - u)| \cdot |t^2 - \phi|W \leq (1 - r(t, \phi)^2)^{1/2} |t^2 - \phi|W \text{ a.e. on } E(\phi).$$

Since $-\log(2 \cos v) \leq u$ a.e. on $E(\phi)^c$, we have $|1 - \exp(iv - u)| \leq 1$ a.e. on $E(\phi)^c$. Hence

$$|(t^2 - 1)W - k| = |t^2 - 1| \cdot |1 - \exp(iv - u)|W \leq |t^2 - 1|W \text{ a.e. on } E(\phi)^c.$$

Since $|k| \leq 2|t^2 - \phi|W$ a.e., k is in H^1 . Hence (i) follows. This completes the proof.

If $\chi_{E(\phi)} \log|\phi - 1|$ is integrable, then it is possible to take an integrable function u in condition (ii). If $r(t, \phi)$ is bounded away from zero, then it is possible to take a bounded function u in (ii).

§ 3. LEFT INVERTIBILITY.

We shall give the form of a weight W such that $\phi P_+ + P_-$ is bounded and left invertible in $L^2(W)$. If W is in (A_2) , then $\phi P_+ + P_-$ is left invertible in $L^2(W)$ if and only if T_ϕ is left invertible in $H^2(W)$.

Definition. For a t in $I(\phi)$ and a ϕ in $L^\infty(m)$, let

$$\begin{aligned} L(t, \phi) = \{ & \ell = u - \bar{v} - \bar{s} - c; \\ & |v| \leq \cos^{-1} r(t, \phi) \text{ a.e., } m\{v = \pi/2\} = 0. \\ & |u| \leq U(t, \phi, v) \text{ a.e. on } E(\phi), \text{ and } -\log(2 \cos v) \leq u \text{ a.e. on } E(\phi)^c. \\ & s \in A(t^2 - \phi), \text{ and } c \text{ is a real constant.} \end{aligned}$$

If $r(t, \phi)$ is bounded away from zero, then $L(t, \phi)$ is a convex subset of BMO.

Theorem 1. *Suppose ϕ is a function in $L^\infty(m)$ such that $m(E(\phi)) > 0$. Suppose ε is a positive constant such that both ε and ε^{-1} belong to $J(\phi)$. For a weight W such that $\log W$ is integrable, the following conditions are equivalent.*

- (i) $\varepsilon \|f\|_W \leq \|(\phi P_+ + P_-)f\|_W \leq \varepsilon^{-1} \|f\|_W$, for all f in $A + \bar{A}_0$.
- (ii) $\varepsilon \leq 1$, $\varepsilon \leq |\phi| \leq \varepsilon^{-1}$ a.e., $m\{\phi = \varepsilon^2\} = m\{\phi = \varepsilon^{-2}\} = 0$ and there exists an ℓ in $L(\varepsilon, \phi) \cap L(\varepsilon^{-1}, \phi)$ such that

$$W = \left(\frac{\varepsilon}{|\varepsilon^2 - 1|} \chi_{E(\phi)^c} + \frac{1}{|\phi - 1|} \chi_{E(\phi)} \right) \exp \ell \text{ a.e.}$$

If $m(E(\phi)^c) > 0$ then $\varepsilon \neq 1$. If W satisfies one of these conditions, then W^{-1} is integrable.

Proof. By Cotlar-Sadosky's theorem [4], it follows from (i) that there exist two functions k, k' in H^1 such that

$$\begin{aligned} |(\varepsilon^2 - \phi)W - k|^2 &\leq (\varepsilon^2 - 1)(\varepsilon^2 - |\phi|^2)W^2 \text{ a.e.}, \\ |(\varepsilon^{-2} - \phi)W - k'|^2 &\leq (\varepsilon^{-2} - 1)(\varepsilon^{-2} - |\phi|^2)W^2 \text{ a.e.} \end{aligned}$$

Since $m(E(\phi)) > 0$, it follows that k and k' are non-zero functions. Suppose $m\{\phi = \varepsilon^2\} > 0$, then $m\{k=0\} > 0$. Since k is in H^1 , we have $k=0$ a.e. (cf.[8], p.76). This contradiction implies $m\{\phi = \varepsilon^2\} = 0$. In the same way we have $m\{\phi = \varepsilon^{-2}\} = 0$. Then

$$(\varepsilon^{\pm 2} - 1)(\varepsilon^{\pm 2} - |\phi|^2) = \{1 - r(\varepsilon^{\pm 1}, \phi)\} |t^2 - \phi|^2 \text{ a.e.}$$

We use Lemma A to complete the proof.

Remark 1. For a function ϕ such that $|\phi|=1$ a.e., we have $r(\varepsilon, \phi) = r(\varepsilon^{-1}, \phi)$ a.e. and hence $U(\varepsilon, \phi, v) = U(\varepsilon^{-1}, \phi, v)$ a.e.. In this case the condition (ii) in the above theorem becomes as follows.

(ii)' There exist three functions u, v, s and a constant c such that

$$W = \left(\frac{\varepsilon}{|\varepsilon^2 - 1|} \chi_{E(\phi)^c} + \frac{1}{|\phi - 1|} \chi_{E(\phi)} \right) \exp(u - \bar{v} - \bar{s} - c) \text{ a.e.},$$

where $|v| \leq \cos^{-1} r(\varepsilon, \phi)$ a.e., $m\{|v| = \pi/2\} = 0$;

$|u| \leq U(\varepsilon, \phi, v)$ a.e. on $E(\phi)$, and

$-\log(2 \cos v) \leq u$ a.e. on $E(\phi)^c$; $s \in A(\varepsilon^2 - \phi) \cap A(\varepsilon^{-2} - \phi)$.

It should be mentioned that if $\phi = -1$ a.e., then the condition (ii)' becomes the Arocena, Cotlar and Sadosky's condition (cf.[1], [3] and [4]). In this case $\phi P_+ + P_- = -P_+ + P_-$ is invertible if and only

if it is bounded. Then $E(-1) = \mathbf{T}$, $r(\varepsilon, -1) = r(\varepsilon^{-1}, -1) = 2\varepsilon/(1+\varepsilon^2)$ a.e., and $A(\varepsilon^2+1) \cap A(\varepsilon^{-2}+1)$ contains a function $s=0$.

Corollary 1. Suppose ϕ is a function in $L^\infty(m)$ such that $|\phi-1|>0$ a.e. and $J(\phi)$ contains a constant 1. For a weight W such that $\log W$ is integrable, the following conditions are equivalent.

- (i) $\phi P_+ + P_-$ is an isometry in $L^2(W)$.
- (ii) $|\phi|=1$ a.e., and there exist an s in $A(1-\phi)$ and a positive constant C such that

$$W = \frac{C}{|\phi-1|} \exp(-\bar{s}) \text{ a.e.}$$

If W satisfies one of these conditions, then W^{-1} is bounded.

Proof. It follows from (i) that

$$\|(\phi P_+ + P_-)f\|_W = \|f\|_W, \text{ for all } f \text{ in } A + \bar{A}_0.$$

This is the case $\varepsilon=1$ in Theorem 1. Hence, $|\phi|=1$ a.e. and there exists an ℓ in $L(1, \phi)$ such that $W = |\phi-1|^{-1} \exp \ell$ a.e.. Since $r(1, \phi) = 1$ a.e., we have

$$L(1, \phi) = \{-\bar{s} - c; s \in A(1-\phi), \text{ and } c \text{ is a real constant}\}.$$

Since $J(\phi)$ contains 1, $|1-\phi| \exp(\bar{s})$ is bounded for some s in $A(1-\phi)$ and hence W^{-1} is bounded. We use Theorem 1 to complete the proof.

Definition. For a function ϕ in $L^\infty(m)$, let $L(\phi, +)$, $L(\phi, -)$ and $L(\phi)$ denote subsets of real measurable functions such that

$$L(\phi, \pm) = \bigcup_{t \in I(\phi, \pm)} L(t, \phi) \text{ and } L(\phi) = L(\phi, +) \cap L(\phi, -).$$

Theorem 2. Suppose ϕ is a function in $L^\infty(m)$ such that $|\phi-1|>0$ a.e.. Suppose there exists a positive constant δ such that $(0, \delta] \cup [\delta^{-1}, \infty)$ is a subset of $J(\phi)$. For a weight W such that $\log W$ is integrable, the following conditions are equivalent.

- (i) $\phi P_+ + P_-$ is bounded and left invertible in $L^2(W)$.
- (ii) ϕ is bounded away from zero and there exists a function ℓ in $L(\phi)$ such that $W = |\phi-1|^{-1} \exp \ell$ a.e..

If W satisfies one of these conditions, then W^{-1} is integrable.

Proof. We shall show that (i) implies (ii). By (i), there exists a positive constant ε such that both ε and ε^{-1} belong to $J(\phi)$ and

$$\varepsilon \|f\|_w \leq \|(\phi P_+ + P_-)f\|_w \leq \varepsilon^{-1} \|f\|_w, \text{ for all } f \text{ in } A + \bar{A}_0.$$

By Theorem 1, there exists an ℓ in $L(\varepsilon, \phi) \cap L(\varepsilon^{-1}, \phi)$ such that $W = |\phi - 1|^{-1} \exp \ell$ a.e.. Since $L(\varepsilon, \phi) \cap L(\varepsilon^{-1}, \phi)$ is a subset of $L(\phi)$, (ii) follows. The converse is also true. This completes the proof.

Proposition 3. *Suppose $|\phi - 1| > 0$ a.e.. Let t and t' be positive constants satisfying $t < t'$. If $J(\phi) = I(\phi)$, then the following statements are true.*

- (1) If t and t' belong to $I(\phi, +)$, then $L(t, \phi)$ is a subset of $L(t', \phi)$ and $r(t', \phi) \leq r(t, \phi)$ a.e..
- (2) If t and t' belong to $I(\phi, -)$, then $L(t', \phi)$ is a subset of $L(t, \phi)$ and $r(t, \phi) \leq r(t', \phi)$ a.e..

Proof. Put $r = r(t, \phi)$ and $r' = r(t', \phi)$, then

$$r'^2 - r^2 = \frac{(t'^2 - t^2)(|\phi|^2 - t'^2 t^2)}{|t'^2 - \phi|^2 |t^2 - \phi|^2} \text{ a.e..}$$

We shall prove (1). Since t and t' belong to $I(\phi, +)$, we have $r' \leq r$ a.e.. Let ℓ be in $L(t, \phi)$ and put $W = |\phi - 1|^{-1} \exp \ell$, then it follows from Lemma A that there exists a k in H^1 such that

$$|(t^2 - \phi)W - k|^2 \leq (t^2 - 1)(t^2 - |\phi|^2)W^2 \text{ a.e..}$$

By Cotlar-Sadosky's theorem [4],

$$\|(\phi P_+ + P_-)f\|_w \leq t \|f\|_w \leq t' \|f\|_w,$$

for all f in $A + \bar{A}_0$. By Cotlar-Sadosky's theorem, there exists a k' in H^1 such that

$$|(t'^2 - \phi)W - k'|^2 \leq (t'^2 - 1)(t'^2 - |\phi|^2)W^2 \text{ a.e..}$$

By Lemma A, there exists an ℓ' in $L(t', \phi)$ such that $W = |\phi - 1|^{-1} \exp \ell'$ a.e. and hence $\ell = \ell'$ a.e.. Thus $L(t, \phi)$ is a subset of $L(t', \phi)$. The proof of (2) is similar to one of (1). This completes the proof.

Proposition 4. *If $J(\phi) = I(\phi)$ and $r(t, \phi)$ is bounded away from zero for all t in $I(\phi)$, then $L(\phi, +)$, $L(\phi, -)$ and $L(\phi)$ are convex subsets of BMO.*

Proof. Let ℓ and ℓ' be in $L(\phi, +)$. There exist t and t' in $I(\phi, +)$ such that ℓ is in $L(t, \phi)$ and ℓ' is in $L(t', \phi)$. Since $r(t, \phi)$ is bounded away from zero, we have $|\phi - 1| > 0$ a.e. and $U(t, \phi, v)$ is in $L_{\infty}^{\bar{r}}(m)$. Since $|\phi - 1| > 0$ a.e. and $J(\phi) = I(\phi)$, it follows from Proposition 3 that the convex combination of ℓ and ℓ' belongs to either $L(t, \phi)$ or $L(t', \phi)$ which is a convex subset of $L(\phi, +)$. Hence $L(\phi, +)$ is a convex subset of BMO. It follows in the similar way that $L(\phi, -)$ is convex and

hence $L(\phi)$ is also convex.

Proposition 5. (1) If ϕ is an outer function in H^∞ , then $J(\phi) \cup \{1\} = I(\phi)$.

(2) If ϕ is a function in $L^\infty_R(m)$ such that $(\text{ess inf } \phi, \text{ess sup } \phi)$ does not contain zero, then $J(\phi) = I(\phi)$.

Proof. We shall prove (1). Let t be any constant in $I(\phi, +)$ not equal to one. Put $\lambda = t^2 - \phi$, then λ is an invertible function in H^∞ since $|\lambda| \geq t^2 - \text{Re } \phi \geq t^2 - \max\{t, 1\} > 0$ a.e. Hence there exist a function f and a constant c such that $\lambda = \exp(f + i\tilde{f} + ic)$ a.e. Put $s = \tilde{f} + c$, then s is in $A(\lambda)$ since $|\lambda| \exp(\tilde{s}) = c'$ for some constant c' . Thus $I(\phi, +)$ is a subset of $J(\phi) \cup \{1\}$. Let t be any constant in $I(\phi, -)$ not equal to one. We may assume that ϕ is bounded away from zero. Put $\lambda = t^2 - \phi$, then λ is an invertible function in H^∞ since $\text{Re}(t^{-1} - \phi^{-1}) \geq 0$ a.e. and $|\lambda| \geq (1-t)(\text{ess inf } |\phi|) > 0$ a.e. Thus $I(\phi, -)$ is a subset of $J(\phi) \cup \{1\}$. Hence $J(\phi) \cup \{1\} = I(\phi)$. We shall prove (2). Let t be any constant in $I(\phi, +)$. Put $\lambda = t^2 - \phi$, then λ is in $L^\infty_R(m)$ and $\lambda \geq 0$ a.e. since $|\phi| \leq t \leq t^2$ a.e. Put $s = \text{Arg } \lambda$, then $s = 0$ a.e. and hence $|\lambda| \exp(\tilde{s})$ is bounded. Thus $I(\phi, +)$ is a subset of $J(\phi)$. Let t be any constant in $I(\phi, -)$. Since $(\text{ess inf } \phi, \text{ess sup } \phi)$ does not contain zero, we have $\phi \geq 0$ a.e. or $\phi \leq 0$ a.e.. If $\phi \geq 0$ a.e., then $\lambda \leq 0$ a.e. since $\phi \geq t \geq t^2$ a.e.. Put $s = \text{Arg } \lambda$, then $s = -\pi$ a.e. and hence $|\lambda| \exp(\tilde{s})$ is bounded. Thus $I(\phi, -)$ is a subset of $J(\phi)$. If $\phi \leq 0$ a.e., then $\lambda \geq 0$ a.e. and hence $I(\phi, -)$ is a subset of $J(\phi)$. Hence $J(\phi) = I(\phi)$. This completes the proof.

For a weight W , $H^2(W)$ (resp. $H^2_0(W)$) denotes the $L^2(W)$ -norm closure of A (resp. A_0). If W is in (A_2) , then T_ϕ is bounded in $H^2(W)$ and $\phi P_+ + P_-$ is bounded in $L^2(W)$.

Proposition 6. Let ϕ be a function in $L^\infty(m)$. For a W in (A_2) , the following conditions are equivalent.

- (i) $\phi P_+ + P_-$ is left invertible in $L^2(W)$.
- (ii) $P_+ \phi P_+ + P_-$ is left invertible in $L^2(W)$.
- (iii) $T_\phi f$ is left invertible in $H^2(W)$.

Proof. Put

$$\begin{aligned} \varepsilon_1 &= \inf\{\|(\phi P_+ + P_-)f\|_W; f \in A + \bar{A}_0, \|f\|_W = 1\}, \\ \varepsilon_2 &= \inf\{\|(P_+ \phi P_+ + P_-)f\|_W; f \in A + \bar{A}_0, \|f\|_W = 1\}, \text{ and} \\ \varepsilon_3 &= \inf\{\|T_\phi f\|_W; f \in A, \|f\|_W = 1\}. \end{aligned}$$

Suppose $\varepsilon_1 > 0$ and let f be any function in $A + \bar{A}_0$ satisfying $\|f\|_W = 1$. Since $P_+ \phi P_+ + P_- = (\phi P_+ + P_-)(I - P_- \phi P_+)$,

$$\|(P_+\phi P_+ + P_-)f\|_W \geq \varepsilon_1 \|(I - P_-\phi P_+)f\|_W \geq \varepsilon_1 \|I + P_-\phi P_+\|_W^{-1},$$

it follows that $\varepsilon_2 \geq \varepsilon_1 \|I + P_-\phi P_+\|_W^{-1} > 0$. Hence (i) implies (ii). Suppose $\varepsilon_2 > 0$ and let f be any function in A satisfying $\|f\|_W = 1$. Since $\|T_\phi f\|_W \geq \|(P_+\phi P_+ + P_-)f\|_W \geq \varepsilon_2$, we have $\varepsilon_3 \geq \varepsilon_2 > 0$. Hence (ii) implies (iii). Suppose $\varepsilon_3 > 0$ and let f be any function in $A + \bar{A}_0$ satisfying $\|f\|_W = 1$. Since $\|P_+\|_W = \|P_-\|_W$ (cf.[14]),

$$\varepsilon_3 \leq \varepsilon_3 (\|P_+ f\|_W + \|P_- f\|_W) \leq \|T_\phi(P_+ f)\|_W + \varepsilon_3 \|P_- f\|_W \leq (1 + \varepsilon_3) \|P_+\|_W \|(P_+\phi P_+ + P_-)f\|_W.$$

We have $\varepsilon_3 \leq \varepsilon_2 (1 + \varepsilon_3) \|P_+\|_W$ and hence $\varepsilon_2 > 0$. Hence (iii) implies (ii). Suppose $\varepsilon_2 > 0$ and let f be any function in $A + \bar{A}_0$ satisfying $\|f\|_W = 1$. Since $\phi P_+ + P_- = (P_+\phi P_+ + P_-)(I + P_-\phi P_+)$,

$$\|(\phi P_+ + P_-)f\|_W \geq \varepsilon_2 \|(I + P_-\phi P_+)f\|_W \geq \varepsilon_2 \|I - P_-\phi P_+\|_W^{-1},$$

we have $\varepsilon_1 \geq \varepsilon_2 \|I - P_-\phi P_+\|_W^{-1} > 0$. Hence (ii) implies (i). This completes the proof.

Proposition 7. Suppose ϕ is a function in $L^\infty_{\mathbb{R}}(m)$ such that $\phi - 1$ is bounded away from zero, and $[\text{ess inf } \phi, \text{ess sup } \phi]$ does not contain zero. If $\phi P_+ + P_-$ is left invertible in $L^2(W)$, then W is in (A_2) .

Proof. Since $[\text{ess inf } \phi, \text{ess sup } \phi]$ does not contain zero and $\phi P_+ + P_-$ is left invertible, it follows that there exists a constant ε in $I(\phi)$ such that ε^2 does not belong to $[\text{ess inf } \phi, \text{ess sup } \phi]$ and

$$\varepsilon \|f\|_W \leq \|(\phi P_+ + P_-)f\|_W, \text{ for all } f \text{ in } A + \bar{A}_0.$$

By Cotlar-Sadosky's theorem, there exists a k in H^1 such that

$$|(\phi - \varepsilon^2)W - k| \leq \{(|\phi|^2 - \varepsilon^2)(1 - \varepsilon^2)\}^{1/2} W \leq \{1 - r(\varepsilon, \phi)^2\}^{1/2} |\phi - \varepsilon^2| W \text{ a.e.}$$

Since $\phi - \varepsilon^2$ and $\phi - 1$ are bounded away from zero, it follows that $r(\varepsilon, \phi)$ is bounded away from zero. Then $\phi - \varepsilon^2 > 0$ a.e. or $\phi - \varepsilon^2 < 0$ a.e.. By Lemma A, $|\phi - \varepsilon^2|W$ is in (A_2) and hence W is in (A_2) . This completes the proof.

Remark 2. Suppose E is a Borel subset of a unit circle. Suppose ℓ is a function in $L^1_{\mathbb{R}}(m)$ such that $\exp \ell$ is integrable, not in (A_2) , $-\log 2 \leq \ell$ a.e. on E^c , and $|\ell| \leq \cosh^{-1}\{(1 + \varepsilon^2)/(2\varepsilon)\}$ a.e. on E . For a constant ε satisfying $0 < \varepsilon \leq 1$, put

$$R(E, \varepsilon) = \{W ; \varepsilon \|f\|_W \leq \|((1 - 2\chi_E)P_+ + P_-)f\|_W \leq \varepsilon^{-1} \|f\|_W, \text{ for all } f \text{ in } A + \bar{A}_0\}.$$

The following statements are then true.

- (a) If $0 < m(E) < 1$, $0 < \varepsilon < 1$ and $W = \{(2\varepsilon)/(1 - \varepsilon^2)\chi_{E^c} + \chi_E\} \exp \ell$, then W is in $R(E, \varepsilon)$, not in (A_2) .
- (b) If $m(E) = 1$, then $(1 - 2\chi_E)P_+ + P_- = -P_+ + P_-$ and hence $R(E, \varepsilon)$ is a subset of (A_2) .

In this section, we have assumed that $\log W$ is integrable. Similar results hold on the assumption that $W > 0$ a.e.. If $m\{W = 0\} > 0$, then the following conditions are equivalent.

- (i) $\phi P_+ + P_-$ is bounded and left invertible in $L^2(W)$.
- (ii) $W = 0$ a.e. on $E(\phi)$, and W has no restriction on $E(\phi)^c$.

§ 4. INVERTIBILITY.

We shall consider the invertibility of operators $\phi P_+ + P_-$ and T_ϕ in weighted spaces. If W is in (A_2) , then $\phi P_+ + P_-$ is invertible in $L^2(W)$ if and only if T_ϕ is invertible in $H^2(W)$. We shall use Rochberg theorem (cf.[16]) to prove Theorem 8.

Theorem 8. *Let ϕ be a function in $L^\infty(m)$. For a W in (A_2) , the following conditions are equivalent.*

- (i) $\phi P_+ + P_-$ is invertible in $L^2(W)$.
- (ii) T_ϕ is invertible in $H^2(W)$.
- (iii) ϕ can be written as

$$\phi = \exp(U + ic - i\tilde{V}) \text{ a.e.}$$

with c a real constant; U a function in $L^\infty_R(m)$; V a real measurable function such that We^V is in (A_2) .

If ϕ and W satisfy one of these conditions, then

$$\|I + P_- \phi P_+\|_W^{-1} \|T_\phi^{-1}\|_W \leq \|(\phi P_+ + P_-)^{-1}\|_W \leq (1 + \|T_\phi^{-1}\|_W) \|P_+\|_W \|I - P_- \phi P_+\|_W.$$

Proof. Rochberg [16] proved (ii) is equivalent to (iii). We shall show that (i) implies (ii). By Proposition 6, (i) implies that T_ϕ is left invertible in $H^2(W)$. Let g be any function in $L^2(W)$. Since $\phi P_+ + P_-$ has a dense range in $L^2(W)$, $T_\phi P_+ = P_+(\phi P_+ + P_-)$ on $A + \bar{A}_0$, and P_+ is bounded in $L^2(W)$, it follows that T_ϕ has a dense range in $H^2(W)$. We shall show that (iii) implies (i) parallel to Rochberg [16]. Let ϕ_1 be a function such that

$$\phi_1 = \exp \frac{1}{2} \{(U + i\tilde{U}) - (V + i\tilde{V})\} \text{ a.e.}$$

and put $\phi_2 = \phi / \phi_1$ then ϕ_1 and $\bar{\phi}_2$ are invertible function in H^p for some $p, p > 1$ such that $|\phi_1|^2 = \exp(U - V)$ a.e. and $|\phi_2|^2 = \exp(U + V)$ a.e.. Define the operator R by

$$Rf = (\phi_1^{-1}P_+ + \phi_2P_-)(\phi_2^{-1}f), f \text{ is in } A + \bar{A}_0.$$

Since $\phi_2^{-1}f$ is in $L^2(We^V) \cap L^p(m)$ for some constant $p, p > 1$ we have

$$\begin{aligned} \|Rf\|_W &\leq \|\phi_1^{-1}P_+(\phi_2^{-1}f)\|_W + \|\phi_2P_-(\phi_2^{-1}f)\|_W \\ &\leq (\exp \|U\|_\infty)^{1/2} (\|P_+(\phi_2^{-1}f)\|_{We^V} + \|P_-(\phi_2^{-1}f)\|_{We^V}) \\ &\leq 2(\exp \|U\|_\infty)^{1/2} \|P_+\|_{We^V} \|\phi_2^{-1}f\|_{We^V} \\ &\leq 2(\exp \|U\|_\infty) \|P_+\|_{We^V} \|f\|_W. \end{aligned}$$

The third inequality holds since We^V is in (A_2) . Thus R extends to a bounded map of $L^2(W)$ to $L^2(W)$. We shall show that for a function f in $A + \bar{A}_0$, $R(\phi P_+ + P_-) = (\phi P_+ + P_-)R = f$. Since $P_+\phi_1P_+ = \phi_1P_+$, $P_-\phi_2^{-1}P_- = \phi_2^{-1}P_-$ and $P_-\phi_1P_+ = P_+\phi_2^{-1}P_- = 0$, we have

$$R(\phi P_+ + P_-)f = (\phi_1^{-1}P_+ + \phi_2P_-)(\phi_2^{-1}(\phi P_+ + P_-)f) = (\phi_1^{-1}P_+ + \phi_2P_-)((\phi_1P_+ + \phi_2^{-1}P_-)f) = f.$$

Since $P_+\phi_1^{-1}P_+ = \phi_1^{-1}P_+$, $P_-\phi_2P_- = \phi_2P_-$, $P_-\phi_1^{-1}P_+ = P_+\phi_2P_- = 0$, we have

$$(\phi P_+ + P_-)Rf = (\phi P_+ + P_-)(\phi_1^{-1}P_+ + \phi_2P_-)(\phi_2^{-1}f) = f.$$

Hence $\phi P_+ + P_-$ has a bounded inverse, namely R . Hence (i) follows. The operator norm inequality follows from the proof of Proposition 6. This completes the proof.

For a W in (A_p) , the necessary and sufficient conditions for T_ϕ to be invertible in $H^p(W)$ was given by Rochberg (cf.[16]). Theorem 8 is the case $p=2$. It is possible to modify this theorem for $p, 1 < p < \infty$.

Proposition 9. *For a weight W in (A_2) , either of the following two conditions imply that $\phi P_+ + P_-$ has a dense range in $L^2(W)$.*

- (a) ϕ is an outer function in H^∞ .
- (b) ϕ is a function in $L^\infty(m)$ such that $(\text{ess inf } \phi, \text{ess sup } \phi)$ does not contain zero.

Proof. Since W is in (A_2) , there exists an invertible function h in H^2 such that $W = |h|^2$ a.e.. Let $(\cdot, \cdot)_W$ denote the inner product in $L^2(W)$. Let g be a function in $L^2(W)$ such that $((\phi P_+ + P_-)f, g)_W = 0$, for all f in $L^2(W)$. Since f_+/h is in $H^2(W)$ and f_-/\bar{h} is in $\bar{H}^2(W)$, we have $(\phi(f_+/h), g)_W = 0$ for all f_+ in A , and $((f_-/\bar{h}), g)_W = 0$ for all f_- in \bar{A}_0 . Hence $\phi\bar{h}g$ is in H_0^2 and $\bar{h}g$ is in H^2 . Put $F = Wg$ and $G = \bar{\chi}\bar{g}\phi W$, then F and G are functions in H^1 and hence FG belongs to $H^{1/2}$.

Suppose (a) holds. Since $(\chi FG)/\phi = W^2|g|^2 \geq 0$ a.e., $(\chi FG)/\phi$ is a function in $H^{1/2}$ which is real and non-negative almost everywhere. Hence there exists a constant C such that $(\chi FG)/\phi = C$ a.e. (cf. [6], p.95). Since ϕ is an outer function, $C=0$. Since ϕ and W are non-zero functions, $g=0$ a.e.. Suppose (b) holds. Since $\chi FG = \phi W^2|g|^2$ a.e. and $(\text{ess inf } \phi, \text{ess sup } \phi)$ does not contain zero, we have $\chi FG \geq 0$ a.e. or $\chi FG \leq 0$ a.e.. Since χFG is in $H^{1/2}$, there exists a constant C such that $\chi FG = C$ a.e.. Hence $g=0$ a.e.. This completes the proof.

Proposition 10. *Suppose ϕ is an outer function in H^∞ not equal to one. Let ε be a positive constant. For a weight W in (A_2) , $\phi P_+ + P_-$ has a dense range in $L^2(W)$ and the following conditions are equivalent.*

- (i) $\varepsilon \|f\|_W \leq \|(\phi P_+ + P_-)f\|_W$, for all f in $A + \bar{A}_0$.
- (ii) $\varepsilon \leq \min\{1, |\phi|\}$ a.e. and there exist a positive constant C and two real functions u, v such that

$$W = \frac{C}{r(\varepsilon, \phi)} \exp(u - v) \text{ a.e.},$$

$$|v| \leq \cos^{-1} r(\varepsilon, \phi) \text{ a.e. and } |u| \leq U(\varepsilon, \phi, v) \text{ a.e.}.$$

Proof. By Cotlar-Sadosky's theorem, it follows from (i) that there exists a k in H^1 such that

$$|(\phi - \varepsilon^2)W - k|^2 \leq W^2(1 - \varepsilon^2)(|\phi|^2 - \varepsilon^2) \text{ a.e.}$$

Put $g = \varepsilon^2 - \phi$, then g is in H^∞ . Put $k = -\varepsilon^{-2}\phi^{-1}$, then k and k^{-1} belong to H^∞ , since ϕ is an outer function and $\varepsilon \leq |\phi|$ a.e.. Let s be any function in $A(\varepsilon^2 - \phi)$. Since $\text{Re } kg \geq 0$ a.e. and

$$\frac{\exp(is - \bar{s})}{|\exp(is - \bar{s})|} = \frac{g}{|g|} \text{ a.e.},$$

there exists a positive constant γ such that $\exp(is - \bar{s}) = \gamma g$ a.e. (cf.[11], Proposition 5). Hence $\bar{s} = -\log|\varepsilon^2 - \phi| + c$ a.e. for some real constant c . We use Lemma A to complete the proof.

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