タイトル	On Weight Functions and Norms of Some Singular Integral Operators
著者	YAMAMOTO, Takanori
引用	北海学園大学学園論集, 121: 1-8
発行日	2004-09-25

On Weight Functions and Norms of Some Singular Integral Operators

Takanori YAMAMOTO

Dedicated to Professor Takahiko Nakazi on the occasion of his 60th birthday

ABSTRACT

Let m be a normalized Lebesgue measure on the unit circle $\mathbf{T}$. Let a and b be bounded m-measurable functions on $\mathbf{T}$, and let w be a positive and integrable function on $\mathbf{T}$. It is well known that the Riesz projection P_+ is bounded on the weighted space $L^2(wdm)$ if and only if w has the Helson-Szegő representation. Let $P_-=I-P_+$, where I denotes the identity operator. In this paper, if the singular integral operator $aP_+ + bP_-$ is bounded and invertible on the weighted space $L^2(wdm)$, then we establish the Helson-Szegő type representation of the weight function w using the operator norms of $aP_+ + bP_-$ and $(aP_+ + bP_-)^{-1}$.

KEYWORDS: Singular integral operator, Riesz projection, Norm, Hardy space, Weight function, Helson-Szegő weight.

MSC (2000): Primary 45E10, 47B35; Secondary 46J15.

1. INTRODUCTION

Let m denote the normalized Lebesgue measure on the unit circle $\mathbf{T}=\{z\,;\,|z|=1\}$. That is, $dm(\zeta)=\dfrac{d\theta}{2\pi}$ for $\zeta=e^{i\theta}$. Throughout this paper, we always assume that the weight function w satisfies $w>0$ a.e. on $\mathbf{T}$, and $w\in L^1(\mathbf{T})=L^1(\mathbf{T}, dm)$. Let $\mathcal{P}$ denote the set of all trigonometric polynomials. Define the Riesz projection P_+ by

$$(P_+f)(e^{it})=\sum_{k\geq 0}\hat{f}(k)e^{ikt}, \quad f\in\mathcal{P},$$

where $\hat{f}(k)$ denotes the k-th Fourier coefficient of f. Let S be the singular integral operator (SIO) defined by

$$(Sf)(\zeta)=\frac{1}{\pi i}\int_{\mathbf{T}}\frac{f(z)}{z-\zeta}dz,$$

where the integral is understood in the sense of Cauchy's principal value (cf. [13], p.11). If f is in L^1, then $(Sf)(\zeta)$ exists for almost everywhere ζ on $\mathbf{T}$, and Sf becomes a measurable function on $\mathbf{T}$. Let $P_-=I-P_+$, where I denotes the identity operator. Then $P_+^2=P_+$, $P_-^2=P_-$, $P_+=\frac{I+S}{2}$, and $P_-=\frac{I-S}{2}$. Since $S=P_+-P_-$, $S^2=I$. The harmonic conjugate function $\tilde{f}$ of f is defined by

$$\tilde{f}(e^{i\theta})=\int_{-\pi}^{\pi}\cot\left(\frac{\theta-t}{2}\right)f(e^{it})\frac{dt}{2\pi},$$

where the integral is understood in the sense of Cauchy's principal value. The Hilbert transform H is defined by $H=-i(P_+-P_-)$. Hence $H=-iS$. Then

$$\begin{aligned}\tilde{f}&=-i(P_+f-P_-f)+i\hat{f}(0)\\&=-i(Sf-\hat{f}(0))\\&=Hf+i\hat{f}(0).\end{aligned}$$

In particular if $\hat{f}(0)=0$, then $\tilde{f}=Hf$. The weighted L^2-norm is defined by

$$\|f\|_w=\|f\|_{L^2(wdm)}=\left(\int_{\mathbf{T}}|f|^2wdm\right)^{1/2}.$$

Let $a,b\in L^\infty(\mathbf{T})=L^\infty(\mathbf{T},dm)$. If aP_++bP_- is a bounded operator on $L^2(wdm)$, then the operator norm of aP_++bP_- is defined by

$$\|aP_++bP_-\|=\|aP_++bP_-\|_{B(L^2(w))}=\sup\{\|(aP_++bP_-)f\|_w\,;f\in L^2(w),\ \|f\|_w=1\}.$$

H. Helson and G. Szegő proved that the Riesz projection P_+ is bounded on $L^2(d\mu)$ if and only if $d\mu=wdm$ is an (HS) measure (cf. [3], [12], [14], [15], [20], [21], [26]). Let H^1 denote the Hardy space. H^1 is a norm-closed subspace of $L^1(\mathbf{T})$. There is a deep extension of the Helson-Szegő approach developed in a series of papers by M. Cotlar, C. Sadosky, R. Arocena and M. Dominguez (cf. [21] Vol.1, p.132, [24], [4], [2], [1], [23], [5], [8], [6], [25], [7]). P. Koosis [16] established the two weights norm inequality for the Hilbert transform on weighted L^2 spaces. In particular, M. Cotlar and C. Sadosky proved the following:

Theorem A. (Cotlar-Sadosky) *For a positive constant M satisfying $M\geq 1$, the following are equivalent:*

(*i*) *the Hilbert transform H is bounded on $L^2(wdm)$ and $\|H\|\leq M$;*

(*ii*) *there exists $h\in H^1$ such that $|w-h|\leq\frac{M^2-1}{M^2+1}w=\left(1-\frac{2}{M^2+1}\right)w$ a.e. on $\mathbf{T}$;*

(*iii*) *there exists* $h \in H^1$ *such that* $|w-h|^2 \leq \left(1-\left(\frac{2M}{M^2+1}\right)^2\right)w^2$ *a.e. on* $\mathbf{T}$;

(*iv*) *there exist real functions* u *and* v *such that* $w=\exp(u+\tilde{v}+const.)$, $\|v\|_\infty \leq \arccos\left(\frac{2M}{M^2+1}\right)=\frac{\pi}{2}-\arcsin\left(\frac{2M}{M^2+1}\right)$ *and* $|u| \leq \cosh^{-1}\left(\frac{M^2+1}{2M}\cos v\right)$ *a.e. on* $\mathbf{T}$,

where $\cosh^{-1}x=\log(x+\sqrt{x^2-1})$.

If the SIO $(aP_+ + bP_-)$ is bounded and invertible on the weighted space $L^2(wdm)$, then we shall consider the explicit form of the (HS) type weights w using functions arccos, arcsin and $\cosh^{-1}$ (cf. [19], [28], [29], [30]). In this paper, Theorems 3 and 4 are the main theorems.

2. MAIN THEOREMS AND COROLLARIES

In this section, we shall give the main theorems and their corollaries. We use Lemmas 1 and 2 to prove the main theorems. The proof of Lemma 1 is essential (cf. [19], [28], [29], [30]), which uses the Cotlar-Sadosky lifting theorem. There are many kinds of proofs of the lifting theorem.

Lemma 1. *For* $a,b \in L^\infty$ *and a positive constant* M *satisfying* $\max(|a|,|b|) \leq M$, *let* $d=\left|\frac{(a-b)M}{M^2-a\bar{b}}\right|$, *then* $\|d\|_\infty \leq 1$, *and the following are equivalent:*

(*i*) *the SIO* $(aP_+ + bP_-)$ *is bounded on* $L^2(wdm)$ *and* $\|aP_+ + bP_-\| \leq M$;

(*ii*) *there exists* $h \in H^1$ *such that* $|(M^2-a\bar{b})w-h|^2 \leq (M^2-|a|^2)(M^2-|b|^2)w^2$ *a.e. on* $\mathbf{T}$;

(*iii*) *there exists* $h \in H^1$ *such that* $|w-h|^2 \leq (1-d^2)w^2$ *a.e. on* $\mathbf{T}$;

(*iv*) *there exist real functions* u *and* v *such that* $w=\exp(u+\tilde{v}+const.)$ *a.e.*, $|v| \leq \arccos d=\frac{\pi}{2}-\arcsin d$ *a.e. and* $|u| \leq \cosh^{-1}\left(\frac{\cos v}{d}\right)$ *a.e. on* $\mathbf{T}$.

The proof of Lemma 1 is similar to the proof of Theorem A, and the proof of the following Lemma 2 is similar to one of Lemma 1.

Lemma 2. *For* $a,b \in L^\infty$ *and a positive constant* N *satisfying* $\min(|a|,|b|) \geq N$, *let* $d=\left|\frac{(a-b)N}{N^2-a\bar{b}}\right|$, *then* $\|d\|_\infty \leq 1$, *and the following are equivalent:*

(*i*) $N\|f\|_w \leq \|(aP_+ + bP_-)f\|_w$ *for every* $f \in \mathcal{P}$;

(*ii*) *there exists* $h \in H^1$ *such that* $|(N^2-a\bar{b})w-h|^2 \leq (N^2-|a|^2)(N^2-|b|^2)w^2$ *a.e. on* $\mathbf{T}$;

(*iii*) *there exists* $h \in H^1$ *such that* $|w-h|^2 \leq (1-d^2)w^2$ *a.e. on* $\mathbf{T}$;

(*iv*) *there exist real functions* u *and* v *such that* $w = \exp(u + \tilde{v} + const.)$ *a.e.*,

$$|v| \leq \arccos d = \frac{\pi}{2} - \arcsin d \text{ a.e. and } |u| \leq \cosh^{-1}\left(\frac{\cos v}{d}\right) \text{ a.e. on } \mathbf{T}.$$

The following Theorems 3 and 4 are the main theorems of this paper. These follow from Lemmas 1 and 2.

Theorem 3. *For* $a,b \in L^\infty$ *and positive constants* M *and* N *satisfying* $|M^2 - a\bar{b}| \cdot |N^2 - a\bar{b}| > 0$ *and* $N \leq \min(|a|,|b|) \leq \max(|a|,|b|) \leq M$, *let* $d = \max\left(\left|\frac{(a-b)M}{M^2 - a\bar{b}}\right|, \left|\frac{(a-b)N}{N^2 - a\bar{b}}\right|\right)$.
Then $\|d\|_\infty \leq 1$, *and the following are equivalent:*

(*i*) $N\|f\|_w \leq \|(aP_+ + bP_-)f\|_w \leq M\|f\|_w$ *for every* $f \in \mathcal{P}$;

(*ii*) *there exists* $h \in H^1$ *such that* $|(M^2 - a\bar{b})w - h|^2 \leq (M^2 - |a|^2)(M^2 - |b|^2)w^2$ *a.e. on* $\mathbf{T}$;

(*iii*) *there exists* $h \in H^1$ *such that* $|w-h|^2 \leq (1-d^2)w^2$ *a.e. on* $\mathbf{T}$;

(*iv*) *there exist real functions* u *and* v *such that* $w = \exp(u + \tilde{v} + const.)$ *a.e.*,

$$|v| \leq \arccos d = \frac{\pi}{2} - \arcsin d \text{ a.e. and } |u| \leq \cosh^{-1}\left(\frac{\cos v}{d}\right) \text{ a.e. on } \mathbf{T}.$$

Theorem 4 follows from Theorem 3. If wdm is the (HS) measure, then there are many articles concerning the invertibility of the SIO $(aP_+ + bP_-)$ on $L^2(wdm)$ (cf. [3], [9], [10], [13], [17], [21], [22]). In Theorems 3 and 4, we do not assume that wdm is the (HS) measure. In Theorem 4, we establish the (HS) type representation of the weight function w using the operator norms of $aP_+ + bP_-$ and $(aP_+ + bP_-)^{-1}$.

Theorem 4. *For* $a,b \in L^\infty$ *such that the SIO* $(aP_+ + bP_-)$ *is bounded and invertible on* $L^2(wdm)$ *and* $\left|\|aP_+ + bP_-\|^2 - a\bar{b}\right| \cdot \left|\frac{1}{\|(aP_+ + bP_-)^{-1}\|^2} - a\bar{b}\right| > 0$ *a.e. on* $\mathbf{T}$, *let*

$$d = \max\left(\left|\frac{(a-b)\|aP_+ + bP_-\|}{\|aP_+ + bP_-\|^2 - a\bar{b}}\right|, \left|\frac{(a-b)\|(aP_+ + bP_-)^{-1}\|}{1 - a\bar{b}\|(aP_+ + bP_-)^{-1}\|^2}\right|\right).$$

Then $\|d\|_\infty \leq 1$, *and there exist real functions* u *and* v *such that* $w = \exp(u + \tilde{v} + const.)$ *a.e.*, $|v| \leq \arccos d = \frac{\pi}{2} - \arcsin d$ *a.e. and* $|u| \leq \cosh^{-1}\left(\frac{\cos v}{d}\right)$ *a.e. on* $\mathbf{T}$.

The following Corollaries 5, 6 and 7 follow from Theorem 3.

Corollary 5. *For distinct and non-zero complex constants a, b and positive constants M, N satisfying* $|M^2-a\bar{b}|\cdot|N^2-a\bar{b}|>0$ *and* $N\le\min(|a|,|b|)\le\max(|a|,|b|)\le M$, *let* $d=\max\left(\left|\frac{(a-b)M}{M^2-a\bar{b}}\right|,\left|\frac{(a-b)N}{N^2-a\bar{b}}\right|\right)$, *then* $d\le1$, *and the following are equivalent:*

(*i*) $N\|f\|_w\le\|(aP_+ + bP_-)f\|_w\le M\|f\|_w$ *for every* $f\in\mathbf{P}$;

(*ii*) *there exist* $h, k\in H^1$ *such that* $|(M^2-a\bar{b})w-h|^2\le(M^2-|a|^2)(M^2-|b|^2)w^2$ *a.e., and* $|(N^2-a\bar{b})w-k|^2\le(N^2-|a|^2)(N^2-|b|^2)w^2$ *a.e. on* $\mathbf{T}$;

(*iii*) *there exists* $h\in H^1$ *such that* $|w-h|^2\le(1-d^2)w^2$ *a.e. on* $\mathbf{T}$;

(*iv*) *there exist real functions* u *and* v *such that* $w=\exp(u+\tilde{v}+const.)$ *a.e.,* $\|v\|_\infty\le\arccos d=\frac{\pi}{2}-\arcsin d$ *and* $|u|\le\cosh^{-1}\left(\frac{\cos v}{d}\right)$ *a.e. on* $\mathbf{T}$.

Corollary 6. *For distinct and non-zero complex constants a, b and a positive constant M satisfying* $\max(|a|,|b|)\le M$, *let* $d=\left|\frac{(a-b)M}{M^2-a\bar{b}}\right|$, *then* $d\le1$, *and the following are equivalent:*

(*i*) *the SIO* $(aP_+ + bP_-)$ *is bounded on* $L^2(wdm)$ *and* $\|aP_+ + bP_-\|\le M$;

(*ii*) *the SIO* $(aP_+ + bP_-)^{-1}=a^{-1}P_+ + b^{-1}P_-$ *is bounded on* $L^2(wdm)$ *and* $\|(aP_+ + bP_-)^{-1}\|\le\frac{M}{|ab|}$;

(*iii*) *there exists* $h\in H^1$ *such that* $|(M^2-a\bar{b})w-h|^2\le(M^2-|a|^2)(M^2-|b|^2)w^2$ *a.e. on* $\mathbf{T}$;

(*iv*) *there exists* $h\in H^1$ *such that* $|w-h|^2\le(1-d^2)w^2$ *a.e. on* $\mathbf{T}$;

(*v*) *there exist real functions* $u, v\in L^\infty$ *such that* $w=\exp(u+\tilde{v}+const.)$ *a.e.,* $\|v\|_\infty\le\arccos d=\frac{\pi}{2}-\arcsin d$ *and* $|u|\le\cosh^{-1}\left(\frac{\cos v}{d}\right)$ *a.e. on* $\mathbf{T}$.

Corollary 7. *For positive constants M and N satisfying* $N\le1\le M$, *let* $\gamma=\min(M,N^{-1})$, $\delta=\max\left(\frac{2}{M^2+1},\frac{2N^2}{N^2+1}\right)$ *and* $d=\max\left(\frac{2M}{M^2+1},\frac{2N}{N^2+1}\right)$. *Then* $\delta=\frac{2}{\gamma^2+1}$, $d=\frac{2\gamma}{\gamma^2+1}$, *and the following are equivalent:*

(*i*) $N\|f\|_w\le\|Hf\|_w\le M\|f\|_w$ *for every* $f\in\mathbf{P}$;

(*ii*) *there exists* $h\in H^1$ *such that* $|w-h|\le(1-\delta)w$ *a.e. on* $\mathbf{T}$;

(*iii*) *there exists* $h\in H^1$ *such that* $|w-h|^2\le(1-d^2)w^2$ *a.e. on* $\mathbf{T}$;

(*iv*) *there exist real functions* $u, v\in L^\infty$ *such that* $w=\exp(u+\tilde{v}+const.)$, $\|v\|_\infty\le\arccos\left(\frac{2\gamma}{\gamma^2+1}\right)=\frac{\pi}{2}-\arcsin\left(\frac{2\gamma}{\gamma^2+1}\right)$ *and* $|u|\le\cosh^{-1}\left(\frac{\gamma^2+1}{2\gamma}\cos v\right)$ *a.e. on* $\mathbf{T}$.

For distinct and non-zero complex constants a and b, the SIO $(aP_+ + bP_-)$ is bounded on $L^2(wdm)$ if and only if $(aP_+ + bP_-)$ is invertible on $L^2(wdm)$ if and only if the Riesz projection P_+ is bounded on $L^2(wdm)$. Then

$$\|(aP_+ + bP_-)^{-1}\| = \|a^{-1}P_+ + b^{-1}P_-\| = \frac{1}{|ab|}\|bP_+ + aP_-\| = \frac{1}{|ab|}\|aP_+ + bP_-\|.$$

By Theorem 4, we have:

Corollary 8. *For distinct and non-zero complex constants a and b, the SIO* $(aP_+ + bP_-)$ *is bounded on* $L^2(wdm)$ *if and only if* $(aP_+ + bP_-)$ *is invertible. Let*

$$d = \left| \frac{(a-b)\|aP_+ + bP_-\|}{\|aP_+ + bP_-\|^2 - a\bar{b}} \right|.$$

Then $0 < d \leq 1$, *and there exist real functions u and v such that* $w = \exp(u + \tilde{v} + const.)$ *a.e.,* $\|v\|_\infty \leq \arccos d = \frac{\pi}{2} - \arcsin d$ *and* $|u| \leq \cosh^{-1}\left(\frac{\cos v}{d}\right)$ *a.e. on* $\mathbf{T}$.

Example. The relation between the norms of the operators H, P_+, P_- on the space $L^2(w)$;

$$\|P_+\| = \|P_-\| = \frac{\|H\| + \|H\|^{-1}}{2}$$

was remarked by Spitkovsky [27]. Then

$$\|H\| = \|S\| = \|P_+ - P_-\| = \|P_+\| + \sqrt{\|P_+\|^2 - 1}.$$

For $\zeta_0 \in \mathbf{T}$, and $-1 < \delta < 1$, let $w(\zeta) = |\zeta - \zeta_0|^\delta$. Then the equality $\|H\| = \cot\frac{\pi(1-|\delta|)}{4}$ was obtained by Krupnik and Verbitsky [18]. Hence $\|P_+\| = \sec\frac{\pi\delta}{2}$.

For complex constants a, b, the formula of the operator norm of $aP_+ + bP_-$ on $L^2(wdm)$ was obtained by Feldman, Krupnik and Markus (cf. [11], [13, Section 13.5], [30]) as the following:

$$\|aP_+ + bP_-\| = \sqrt{\gamma + \left(\frac{|a|+|b|}{2}\right)^2} + \sqrt{\gamma + \left(\frac{|a|-|b|}{2}\right)^2},$$

where

$$\gamma = \left|\frac{a-b}{2}\right|^2 (\|P_+\|^2 - 1).$$

If $\gamma = 0$, then the right hand side equals to $\max(|a|,|b|)$.

For example, suppose $w(\zeta)=|\zeta-1|^{1/2}$. Then $\|P_+\|=\sec\frac{\pi}{4}=\sqrt{2}$, and $\gamma=\left|\frac{a-b}{2}\right|^2$. Let

$$c=\|aP_++bP_-\|=\sqrt{\left|\frac{a-b}{2}\right|^2+\left(\frac{|a|+|b|}{2}\right)^2}+\sqrt{\left|\frac{a-b}{2}\right|^2+\left(\frac{|a|-|b|}{2}\right)^2},$$

and let $d=\left|\frac{(a-b)c}{c^2-a\bar{b}}\right|$. Then $d\leq 1$, and for every distinct complex constants a and b, we have $d=\frac{1}{\sqrt{2}}$ by the calculation, and $w(\zeta)=|\zeta-1|^{1/2}$ has the Helson-Szegő representation:

there exist real functions u *and* v *such that* $w=\exp(u+\tilde{v}+const.)$ *a.e.*, $\|v\|_\infty\leq\arccos d=\frac{\pi}{4}$ *and* $|u|\leq\cosh^{-1}\left(\frac{\cos v}{d}\right)=\cosh^{-1}(\sqrt{2}\cos v)$ *a.e. on* $\mathbf{T}$.

References

[1] R. Arocena, A refinement of the Helson-Szegő theorem and the determination of the extremal measures, *Studia Math.* **71** (1981/1982), 203-221.

[2] R. Arocena, M. Cotlar and C. Sadosky, Weighted inequalities in L^2 and lifting properties, *Mathematical analysis and Applications*, 7A, pp.95-128, Academic Press, New York, London, 1981.

[3] A. Böttcher and B. Silbermann, *Analysis of Toeplitz operators*, Akademie-Verlag, Berlin, and Springer-Verlag, 1990.

[4] M. Cotlar and C. Sadosky, On the Helson-Szegő theorem and a related class of modified Toeplitz kernels, pp.383-407, *Harmonic analysis in Euclidean spaces* (Williamstown, MA, 1978), Part 1, eds. G. Weiss and S. Wainger, Proc. Symp. Pure Math. 35, Amer. Math. Soc., Providence, 1979.

[5] M. Cotlar and C. Sadosky, Toeplitz liftings of Hankel forms, pp.22-43, *Function spaces and applications* (Lund, 1986), Lect. Notes Math. 1302, Springer-Verlag, Berlin and New York, 1988.

[6] M. Cotlar and C. Sadosky, Weakly positive matrix measures, generalized Toeplitz forms, and their applications to Hankel and Hilbert transform operators, pp.93-120, *Operator Theory: Adv. and Appl.* (Basel, Birkhäuser), vol.58, 1992.

[7] M. Dominguez, Interpolation and prediction problems for connected compact abelian groups, *Integral Equations Operator Theory* **40** (2001), 212-230.

[8] M. Dominguez, Weighted inequalities for the Hilbert transform and the adjoint operator in the continuous case, *Studia Math.* **95** (1990), 229-236.

[9] R.G. Douglas, *Banach algebra techniques in operator theory (2nd ed.)*, Springer-Verlag, New York, Berlin, 1998.

[10] V.B. Dybin and S.M. Grudsky, *Introduction to the theory of Toeplitz operators with infinite index*, Birkhäuser-Verlag, Basel, 2002.

[11] I. Feldman, N. Ya. Krupnik and A. Markus, On the norm of polynomials of two adjoint projections, *Integral Equations Operator Theory* **14** (1991), 69-90.

[12] J.B. Garnett, *Bounded analytic functions*, Academic Press, New York, 1981.

[13] I. Gohberg and N.Ya. Krupnik, *One-dimensional linear singular integral equations*, "Shtiintsa", Kishinev, 1973 (Russian); English transl.: Vol.I and II, Birkhäuser-Verlag, Basel, 1992.

[14] H. Helson and G. Szegő, A problem in prediction theory, *Ann. Mat. Pura. Appl.* **51** (1960), 107-138.

[15] P. Koosis, *Introduction to H^p spaces (2nd ed.)*, Cambridge Univ. Press, 1998.

[16] P. Koosis, Weighted quadratic means of Hilbert transforms, *Duke Math. J.* **38** (1971), 609-634.

[17] N.Ya. Krupnik, *Banach algebras with symbol and singular integral operators*, "Shtiintsa", Kishinev, 1984 (Russian); English transl.: Birkhäuser-Verlag, Basel, 1987.

[18] N.Ya. Krupnik and I. E. Verbitsky, Exact constants in the theorem of K.I. Babenko and B.V. Khvedelidze on the boundedness of singular operators (Russian), *Soobshzh. AN Gruz. SSR* **85** (1977), no.1, 21-24.

[19] T. Nakazi and T. Yamamoto, Some singular integral operators and Helson-Szegő measures, *J. Funct. Analysis* **88** (1990), 366-384.

[20] N.K. Nikolski, *Treatise on the shift operator*, Springer-Verlag, Berlin, 1986.

[21] N.K. Nikolski, *Operators, functions, and systems*, Vol.1 and 2, Math. Surveys and Monographs 92 and 93, Amer. Math. Soc., Providence, 2002.

[22] R. Rochberg, Toeplitz operators on weighted H^p spaces, *Indiana Univ. Math. J.* **26** (1977), 291-298.

[23] C. Sadosky, Some applications of majorized Toeplitz kernels, pp.581-626, *Topics in Modern Harmonic Analysis*, Proc. Seminar Torino and Milano (May-June 1982), Vol.II, Inst. Naz. Alta Matematica F. Severi, Roma, 1983.

[24] C. Sadosky, The mathematical contributions of Mischa Cotlar since 1955, *Analysis and Partial Differential Equations*, pp.715-742, Dekker, New York, 1990.

[25] C. Sadosky, Liftings of kernels shift-invariant in scattering theory, pp.303-336, *Holomorphic spaces*, eds. Sh. Axler, J. McCarthy and D. Sarason, MSRI Publications 33, Cambridge Univ. Press, 1998.

[26] D. Sarason, *Function theory on the unit circle*, Virginia Polytechnic Institute and State Univ., Blacksburg, VA, 1979.

[27] I.M. Spitkovsky, On partial indices of continuous matrix-valued functions, *Soviet Math. Doklady* **17** (1976), 1155-1159.

[28] T. Yamamoto, On the generalization of the theorem of Helson and Szegő, *Hokkaido Math. J.* **14** (1985), 1-11.

[29] T. Yamamoto, On weighted norm inequalities in L^2 on the unit circle, *Journal of Hokkai-Gakuen University* **52** (1985), 13-19.

[30] T. Yamamoto, Boundedness of some singular integral operators in weighted L^2 spaces, *J. Operator Theory* **32** (1994), 243-254.